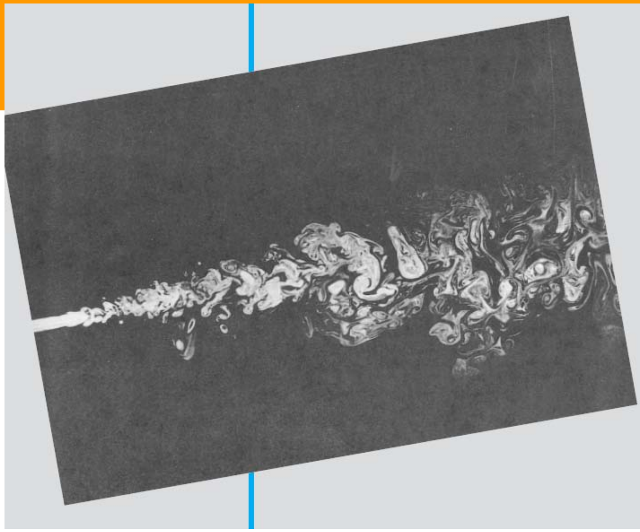
TURBULENT FLOW

A BEGINNER'S APPROACH



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CONTENTS

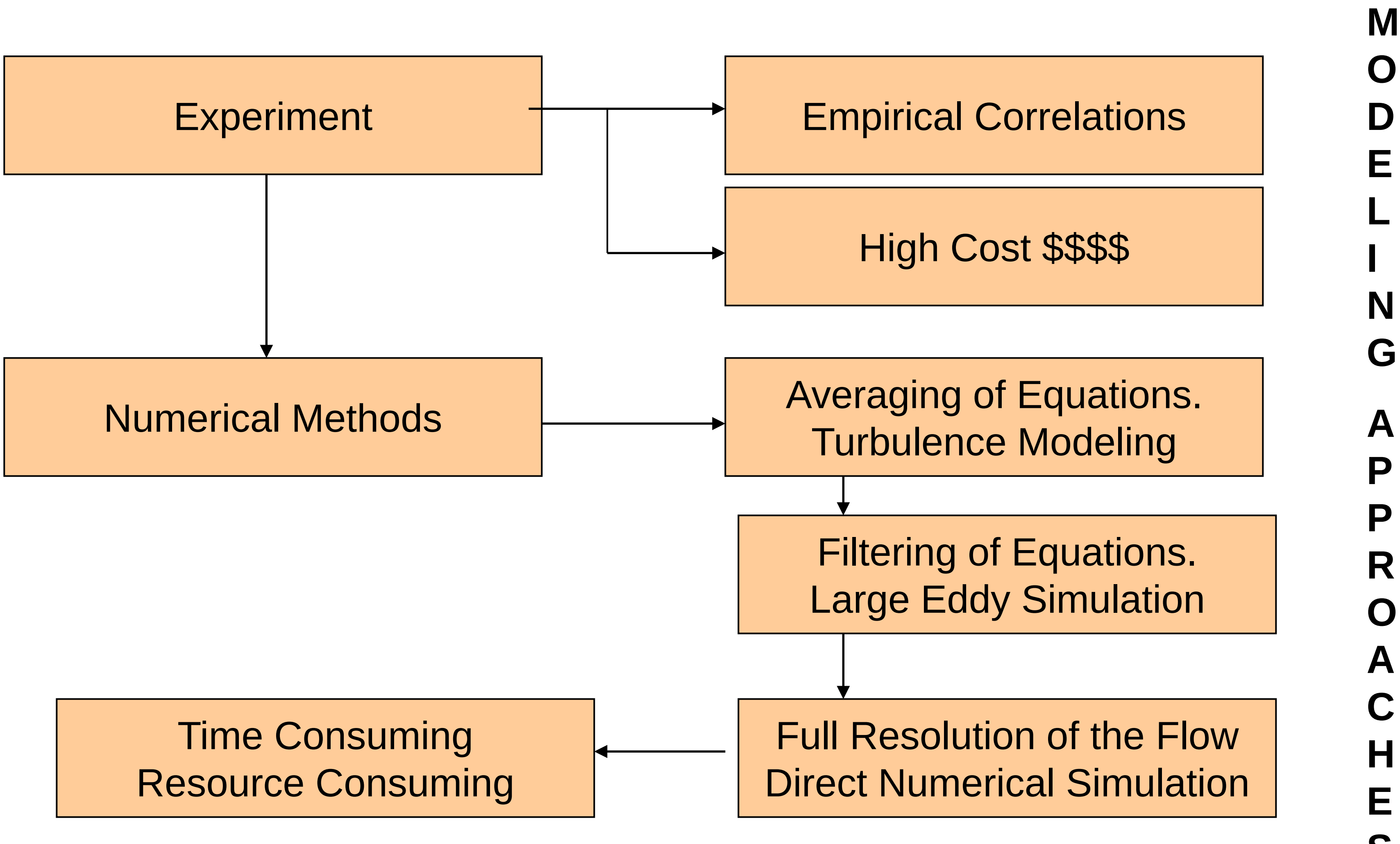
INTRODUCTION

- Most flows encountered in engineering practice are Turbulent
- Turbulent Flows are characterized by a fluctuating velocity field (in both position and time). We say that the velocity field is Random.
- Turbulence highly enhances the rates of mixing of momentum, heat etc…

INTRODUCTION

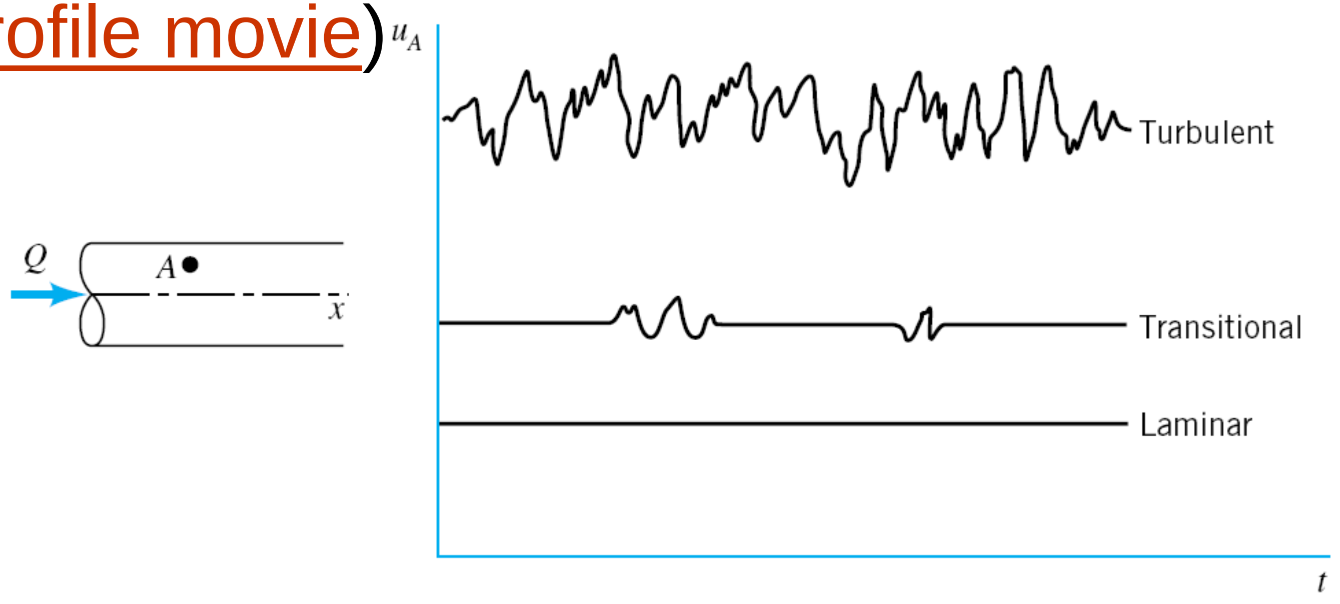
- The motivations to study turbulent flows are summarized as follows:
  - The vast majority of flows is turbulent
  - The transport and mixing of matter, momentum, and heat in turbulent flows is of great practical importance
  - Turbulence enhances the rates of the above processes
- The basic approaches to investigating turbulent flows are shown in the following chart

INTRODUCTION



RANDOM PROCESSES

- Consider a typical pipe flow where the instantaneous velocity is to be measured as a function of time. (see dye movie – see velocity profile movie)



RANDOM PROCESSES

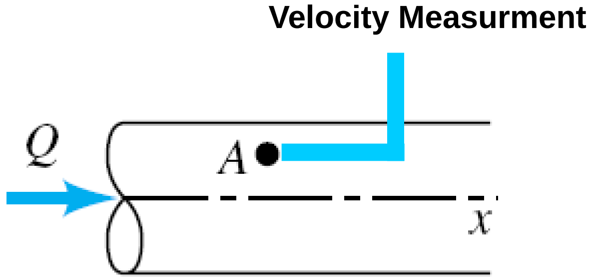
- It is clear that the velocity is fluctuating with time
- We say that the velocity field is RANDOM
- What does that mean?
- Why is it so?

RANDOM PROCESSES

- Consider the pipe flow experiment that is to be repeated several times (say 395804735 times) under the same set of conditions:
  - On the same planet
  - In the same country
  - In the same lab
  - Using the same apparatus
  - Supervised by the same person

RANDOM PROCESSES

- We are interested in measuring the *streamwise* velocity component at a given point $\boldsymbol{A(x_1,y_1,z_1)}$ & at a given time during the experiment: $\boldsymbol{U(A,t_1)}$



RANDOM PROCESSES

- Consider the event denoted by E such that:

$$E = \left\{ U(A, t_1) < 10 \, {}^{m}\!/_{s} \right\}$$

- Three cases take place:
  - If **E** inevitably occurs, then **E** is certain
  - If **E** cannot occur, then **E** is impossible
  - Another possibility is that **E** may but need not occur, then **E** is **Random**

RANDOM PROCESSES

- The word **Random** does not hold any sophisticated significance as it is usually assigned
- The event **E** is random means only that it may or may not occur
- And this answers our first question; i.e. the meaning of a random variable

RANDOM PROCESSES

- We can now check the results of our experiment. After 40 repetitions of the experiment the measured velocity was recorded as shown below:

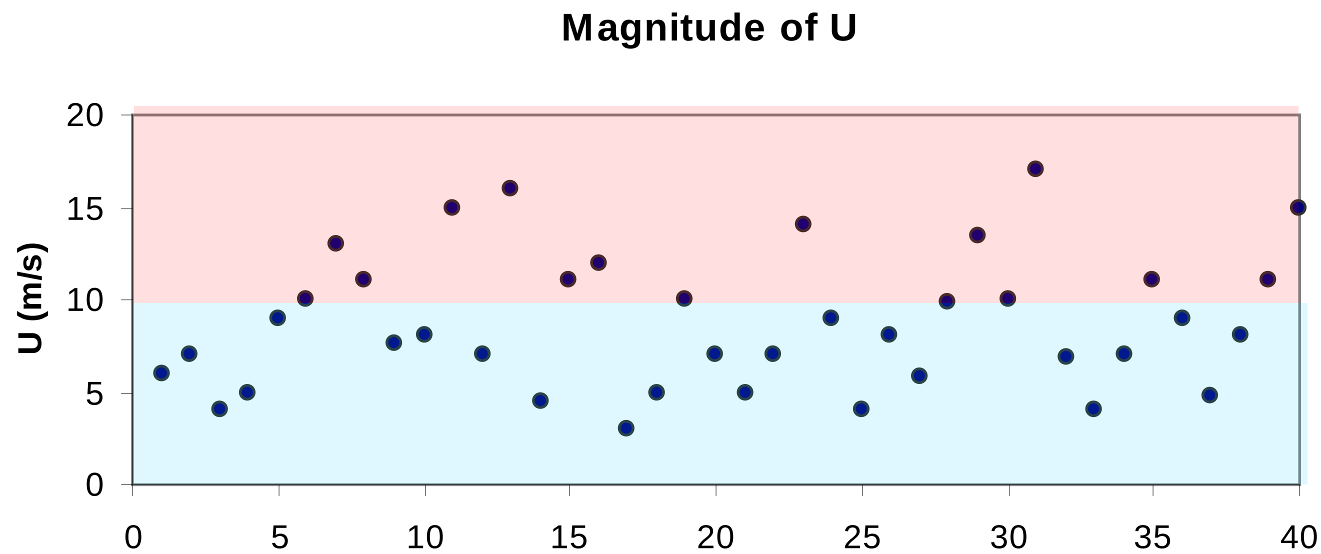


Figure 1.1: Magnitude of U as a function of Experiment Number

RANDOM PROCESSES

- Okay. Why is the velocity field random in a turbulent flow?

RANDOM PROCESSES

- Obviously, if there were no determinism (at least in the underlying structure of the universe), why would we do science in the first place? As science aims @ predicting the behavior of physical phenomena…
- To answer the question, we have to resort to the theory of dynamical systems. After all, the governing equations of fluid mechanics represent a set of dynamical equations.
- Dynamical systems are systems that are extremely sensitive to minute changes in the surrounding environment, i.e. **initial & boundary conditions**

RANDOM PROCESSES

- An illustrative example is the **Lorenz** dynamic system.
- Consider the following set of ODEs

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = \rho x - y - xz$$

$$\dot{z} = -\beta z + xy$$

Assume we fix $\sigma = 10$ and $\beta = 8/3$ and vary ρ

RANDOM PROCESSES

- We are interested in solving this system for two different **acute** initial conditions.

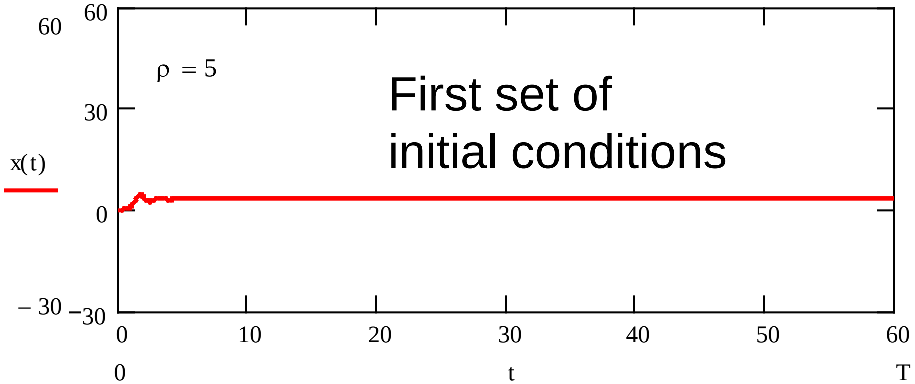
$$[x(0),y(0),z(0)]=[0.1,0.1,0.1] \text{ and}$$

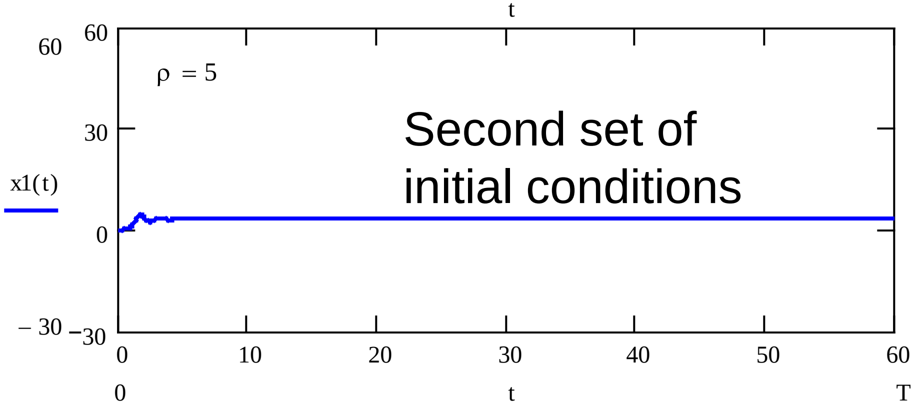
$$[x_1(0),y(0),z(0)]=[0.1000001,0.1,0.1]$$

- Obviously, the two solutions for the different initial conditions should be very close since the initial conditions are approximately the same…This is what we expect. Let us see the results

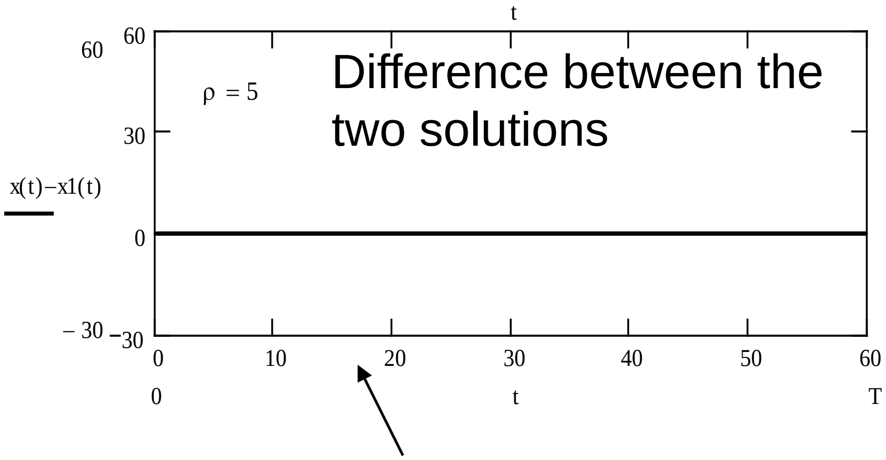
RANDOM PROCESSES

- The solution is easy to get numerically and is as follows for various values of ρ





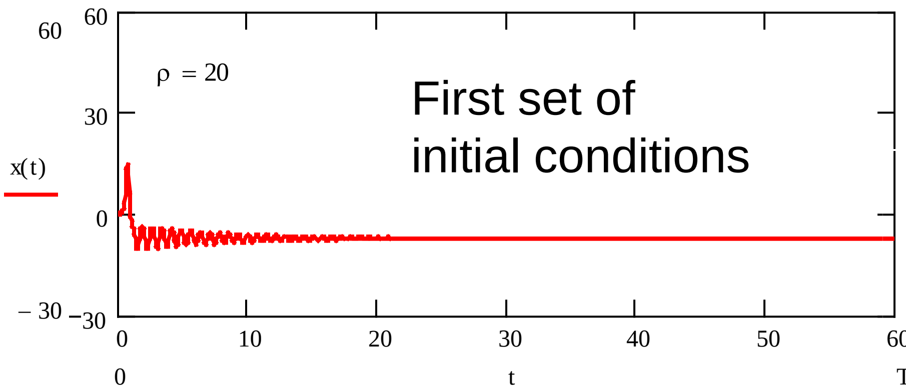
ρ = 5

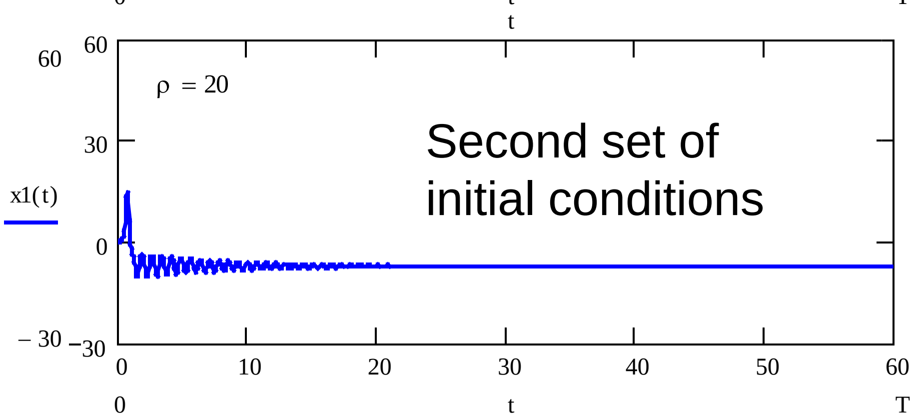


Difference between the two solutions

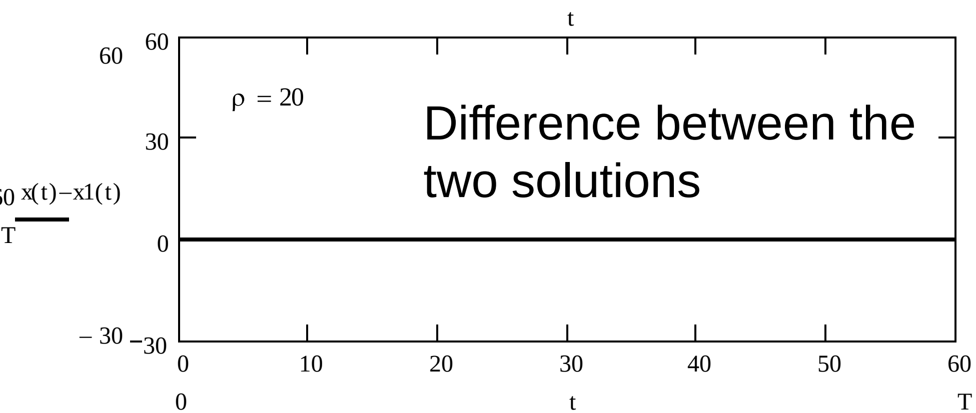
RANDOM PROCESSES

- Nothing much happens for some values of ρ



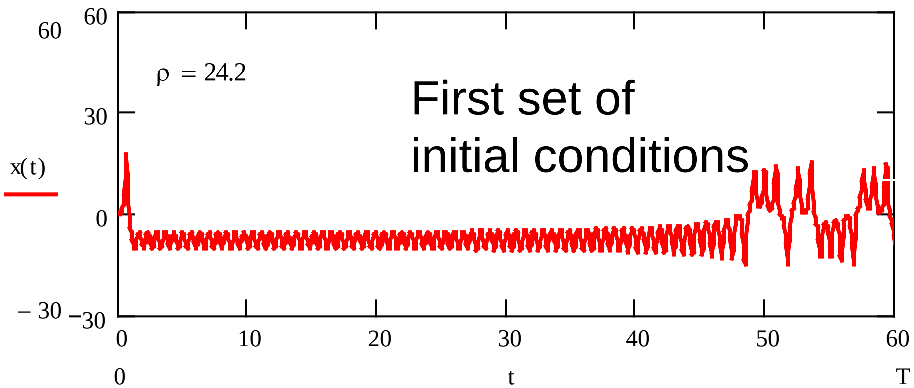


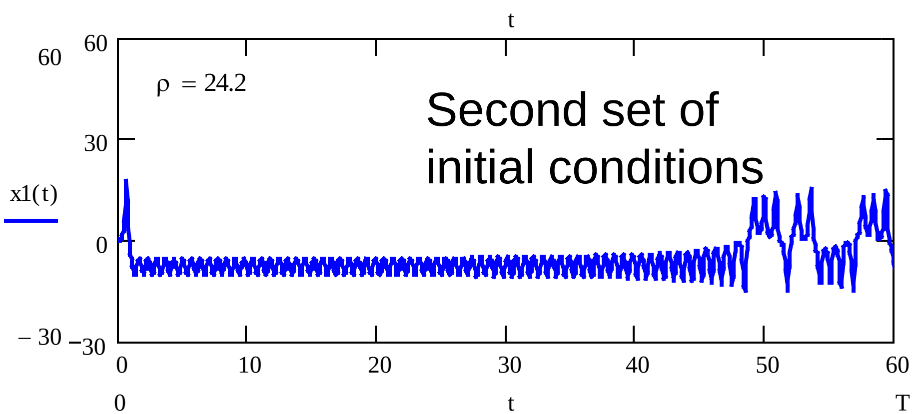
ρ = 20



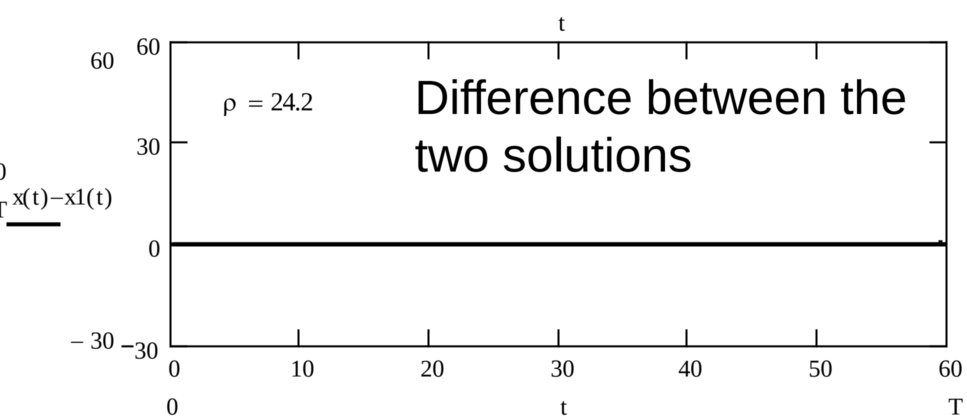
RANDOM PROCESSES

- What is so special about ρ **= *24.2***



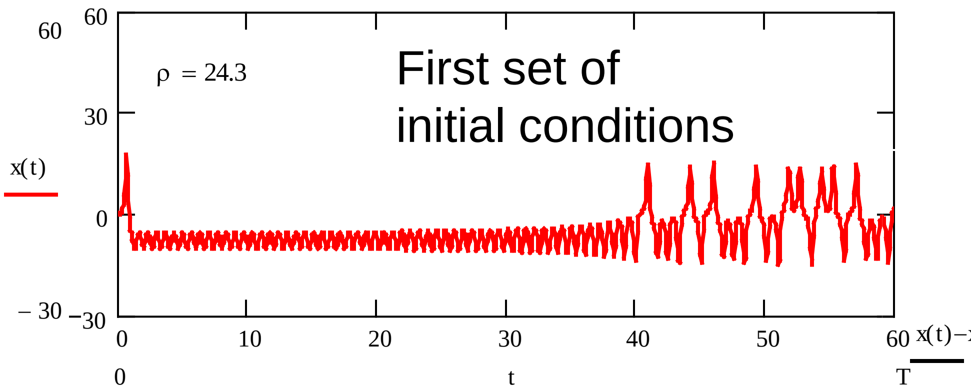


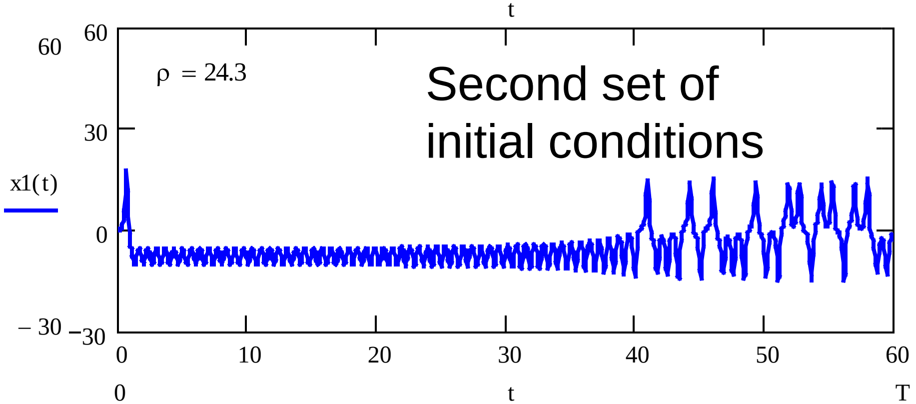
ρ= 24.2



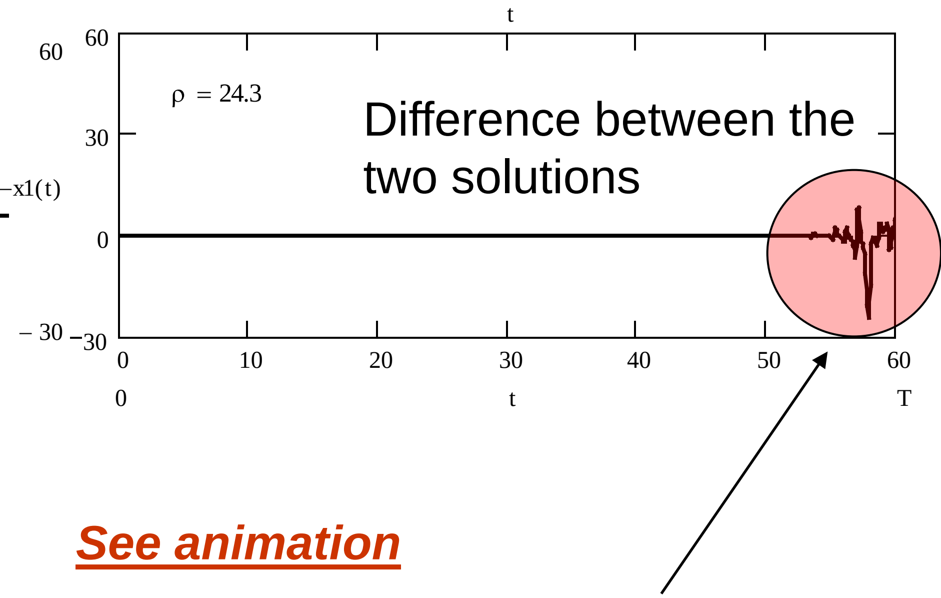
RANDOM PROCESSES

- ***ρ = 24.2 denotes the onset of sensitivity!!!***





ρ = 24.3



See animation

RANDOM PROCESSES

- As we can see, the figures show the sensitivity of the system. In fact, for a critical value of the coefficients (fix σ, β) ρ =24.3, the system becomes highly sensitive to small perturbations.
- This coefficient corresponds to the Reynolds number in fluid flow. Beyond a critical value, the flow becomes extremely sensitive to any minute disturbance whether from the boundary or from the internal structure of the flow (an eddy)

RANDOM PROCESSES

- Conclusion:
  - A theory that predicts exact values of the velocity field is definitely prone to be wrong. Such a theory should aim at predicting the probability of occurrence of certain events

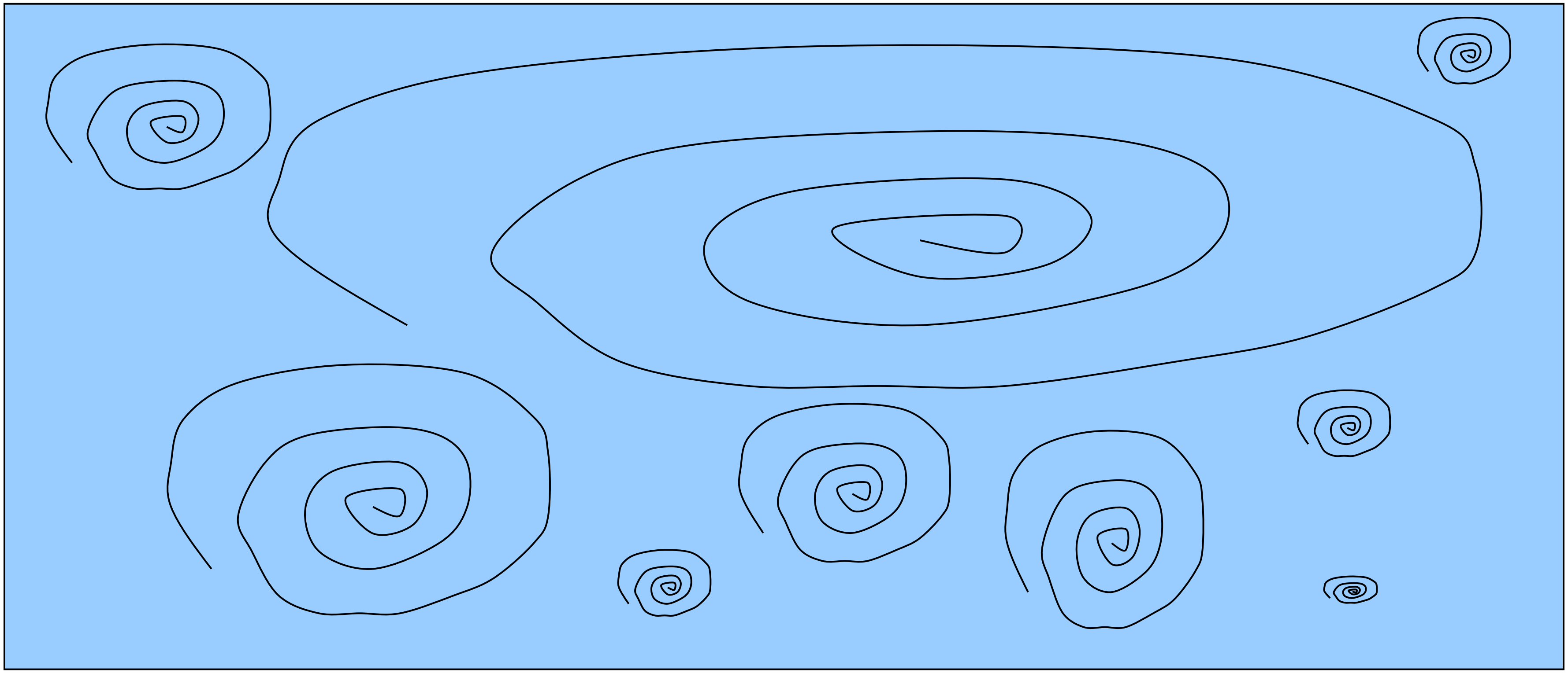
THE ENERGY CASCADE

- One of the first attacks to theoretically tackle the problem of turbulence was done by Richardson in 1922
- His hypothesis is called the Energy Cascade Mechanism & is now one of the fundamental physical models underlying any approach to solving turbulent flow problems
- Turbulence can be considered to be composed of eddies of different sizes

THE ENERGY CASCADE

- An eddy is considered to be a turbulent motion localized within a region of size **l**
- These sizes range from the Flow length scale ***L*** to the smallest eddies.
- Each eddy of length size **l** has a characteristic velocity **u(l)** and timescale τ**(l) = l /** u **(l)**
- The largest eddies have length scales comparable to ***L***

THE ENERGY CASCADE

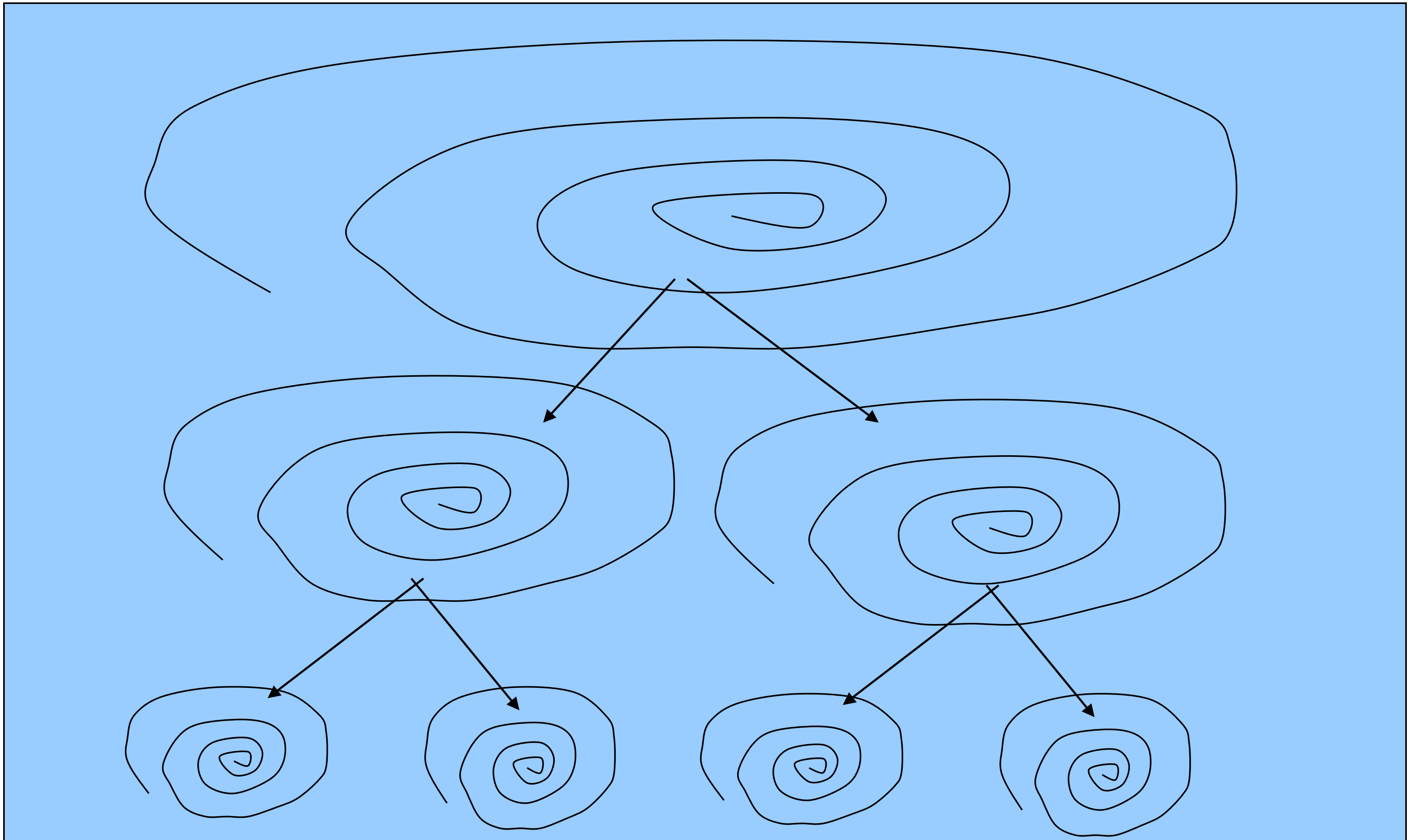


Movie

THE ENERGY CASCADE

- Each eddy has a Reynolds number
- For large eddies, Re is large, i.e. viscous effects are negligible.
- The idea is that the large eddies are unstable and break up transferring energy to the smaller eddies.
- The smaller eddies undergo the same process and so on

THE ENERGY CASCADE



THE ENERGY CASCADE

- This energy cascade continues until the Reynolds number is sufficiently small that energy is dissipated by viscous effects: the eddy motion is stable, and molecular viscosity is responsible for dissipation.

Big whorls have little whorls,
which feed on their velocity;
and little whorls have lesser whorls,
and so on to viscosity

THE KOLMOGOROV HYPOTHESES

- What is the size of the smallest eddies???
- The KOLMOGOROV Hypotheses address this fundamental question along with many others
- There are three propositions that Kolmogorov made:
  - ***The hypothesis of local isotropy***
  - ***First similarity***
  - ***Second similarity***

THE KOLMOGOROV HYPOTHESES

- We first introduce some useful scales
  - Denote by L_0 *the length scale of an eddy comparable in size to the flow geometry*
  - *Denote by* $L_{EI} \approx \frac{1}{6} L_0$ *the demarcation scale between the largest ($L > L_{EI}$) eddies and the smallest eddies ($L < L_{EI}$)*

THE KOLMOGOROV HYPOTHESES

- Kolmogorov's hypothesis of local isotropy:

  At sufficiently high Reynolds number, the small-scale motions ($L < L_0$) are statistically isotropic.

- As the energy passes down the cascade, all information about the directional properties of the large eddies (determined by the flow geometry) is also lost. As a consequence, the small enough eddies have a somehow universal character, independent of the flow geometry

THE KOLMOGOROV HYPOTHESES

- Kolmogorov's first similarity hypothesis:

  In every turbulent flow at sufficiently high Reynolds number, the statistics of the small-scale motions ($L < L_{EI}$) have a universal form that is uniquely determined by ν and ε.

- Just as the directional properties are lost in the energy cascade, all information about the boundaries of the flow is also lost, therefore, the important processes responsible for dissipation are determined uniquely by ν and ε

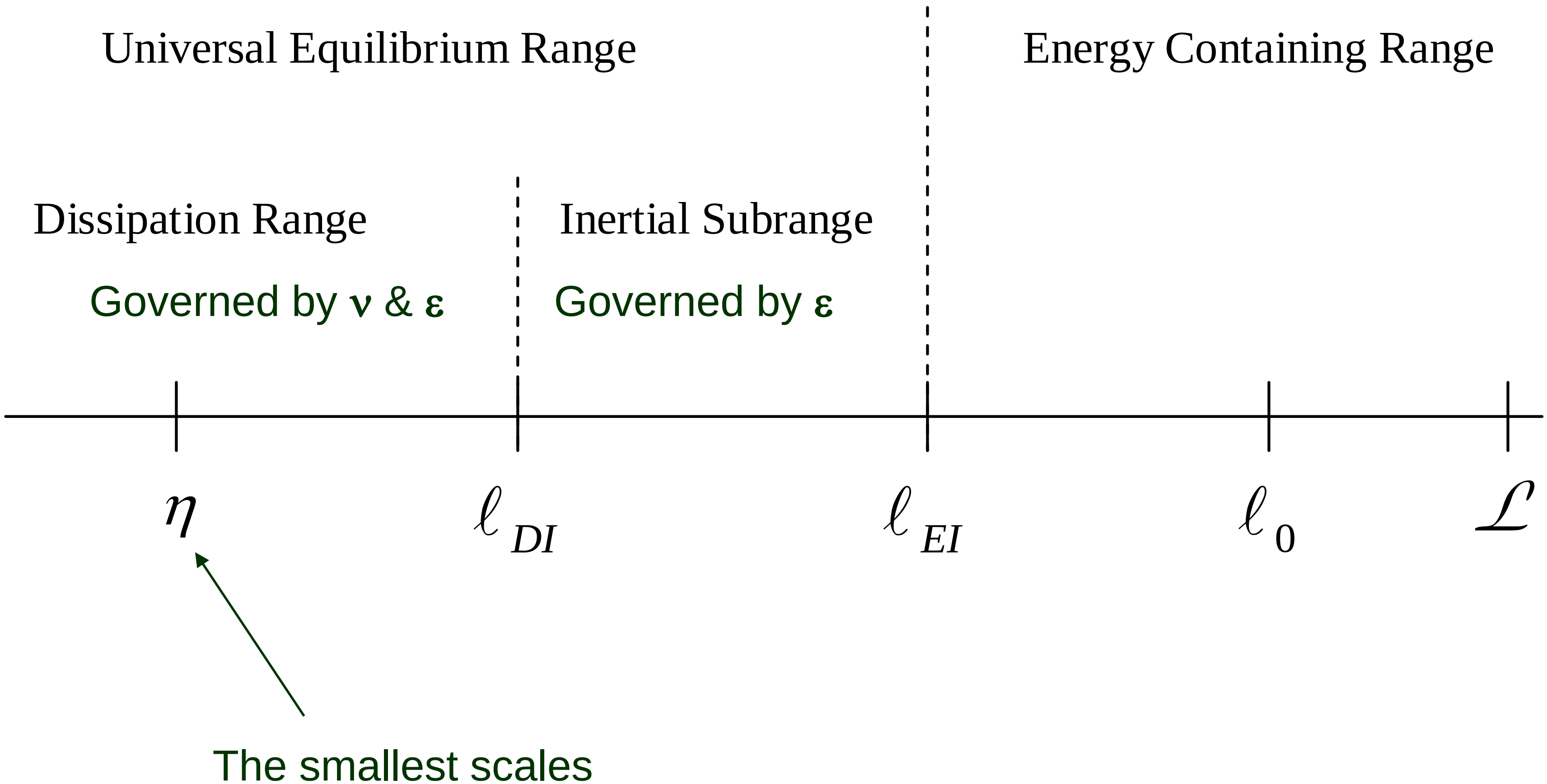
THE KOLMOGOROV HYPOTHESES

- Kolmogorov's second similarity hypothesis:

  In every turbulent flow at sufficiently high Reynolds number, there exists a range of scales whose properties are uniquely determined by ε

- The three hypothesis are shown in the sketch to follow

THE KOLMOGOROV HYPOTHESES



THE CLOSURE PROBLEM

- **Mass Conservation:**

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial \rho u_i}{\partial x_i} = 0$$

- **Momentum Conservation:**

$$\underbrace{\frac{\partial \rho u_i}{\partial \tau}}_{\text{Accum.}} + \underbrace{\frac{\partial \rho u_j u_i}{\partial x_j}}_{\text{Convection}} = \underbrace{\frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j} \right)}_{\text{Diffusion}} \underbrace{- \frac{\partial p}{\partial x_i} + \rho g_i + s_{ui}}_{Source}$$

THE CLOSURE PROBLEM

- The infinitude of scales in a turbulent flow renders the flow equations impossible to tackle mathematically in that the velocity field represents a random variable
- A clever proposition, addressed by Reynolds reduces the numbers of scales from infinity to 1 or 2
- After all, for all practical purposes, we are only interested in the mean flow and in the mean fluid properties

THE CLOSURE PROBLEM

- **Problems**: Elliptic & Non-Linear Equations; Pressure-Velocity Coupling & Temperature-Velocity Coupling; Turbulence: Unsteady, 3-D, Chaotic, Diffusive, Dissipative, Intermittent, ...; **Infinite Number of Scales (Grid $\propto$ Re$^{9/4}$),** Etc.
- **Solution**: Reduce the number of scales $\Rightarrow$ Reynolds Decomposition.

$$\varphi \equiv \Phi + \varphi'$$

Turbulence: RANS

- **Result**: **Reynolds Averaged Navier-Stokes (RANS) Equations**.

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial \rho U_i}{\partial x_i} = 0$$

$$\frac{\partial \rho U_i}{\partial \tau} + \frac{\partial \rho U_j U_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\mu \frac{\partial U_i}{\partial x_j} - \rho \overline{u'_i u'_j} \right) - \frac{\partial P}{\partial x_i} + S_{ui}$$

- **Closure Problem**: Turbulent Stresses $(\rho \overline{u'_i u'_j})$ are unknowns $\Rightarrow$ **Turbulence Models**.

THE CLOSURE PROBLEM

- One can suggest to write transport equations for the reynolds stresses, however, one would end up with triple correlations…and so forth…
- The idea is that we have to stop somewhere and model the correlation using physical arguments

THE CLOSURE PROBLEM

$$\frac{\partial \rho \overline{u'_i u'_j}}{\partial \tau} + \frac{\partial \rho U_k \overline{u'_i u'_j}}{\partial x_k} = -\left(\rho \overline{u'_i u'_k} \frac{\partial U_j}{\partial x_k} + \rho \overline{u'_j u'_k} \frac{\partial U_i}{\partial x_k} \right) \cdots\cdots\cdots (P_{ij})$$

$$-\left(\rho \beta g_i \overline{u'_j t} + \rho \beta g_j \overline{u'_i t} \right) \cdots\cdots\cdots\cdots\cdots\cdots (G_{ij})$$

$$\left. \begin{array}{l} -\dfrac{\partial}{\partial x_k}\left(\rho \overline{u'_i u'_j u'_k} + \delta_{ik} \overline{p' u'_j} + \delta_{jk} \overline{p' u'_i} \right) \\ +\dfrac{\partial}{\partial x_k}\left(\mu \dfrac{\partial \overline{u'_i u'_j}}{\partial x_k} + \mu \overline{u_i \dfrac{\partial u'_k}{\partial x_j}} + \mu \overline{u_j \dfrac{\partial u'_k}{\partial x_i}} \right) \end{array} \right\} (D_{ij})$$

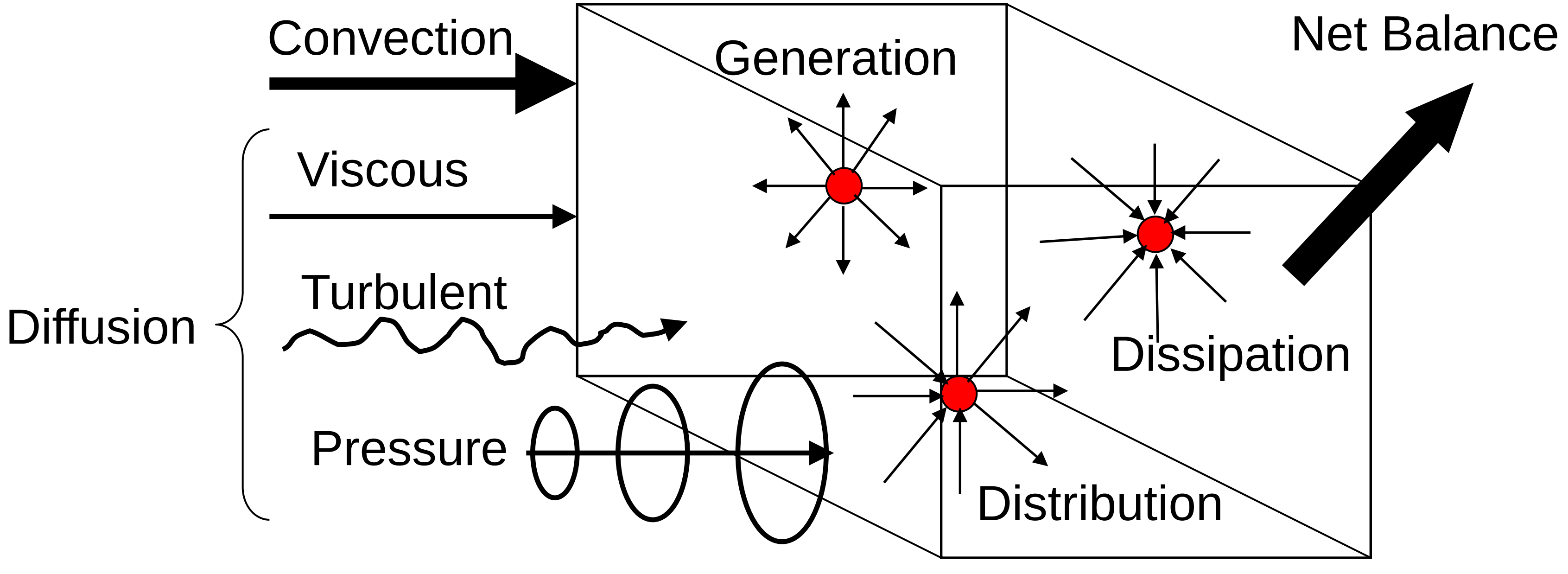
$$+\left(\overline{p' \frac{\partial u'_i}{\partial x_j}} + \overline{p' \frac{\partial u'_j}{\partial x_i}} \right) \cdots\cdots\cdots\cdots\cdots\cdots\cdots (\Phi_{ij})$$

$$-2\mu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} - \mu \left(\overline{\frac{\partial u'_j}{\partial x_k} \frac{\partial u'_k}{\partial x_i}} + \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_k}{\partial x_j}} \right) (\varepsilon_{ij})$$

Example
Of transport
Equation
For the
Turbulent
Stress

THE CLOSURE PROBLEM

- Physical mechanisms governing the change of turbulent stresses & fluxes.



THE CLOSURE PROBLEM

- **Unknown Correlations:**

  -) Turbulent & Pressure Diffusion (D_{ij}, D_{it})

  -) Pressure-Strain Correlations (Φ_{ij}, Φ_{it})

  -) Dissipation Rates (ε_{ij}, ε_{it})

  -) Possible Sources (S_{it})

  Modeling is required in order to close the governing equations.

- **Turbulence Models:**

  -) Algebraic (zero-equation) models: Mixing length, etc.

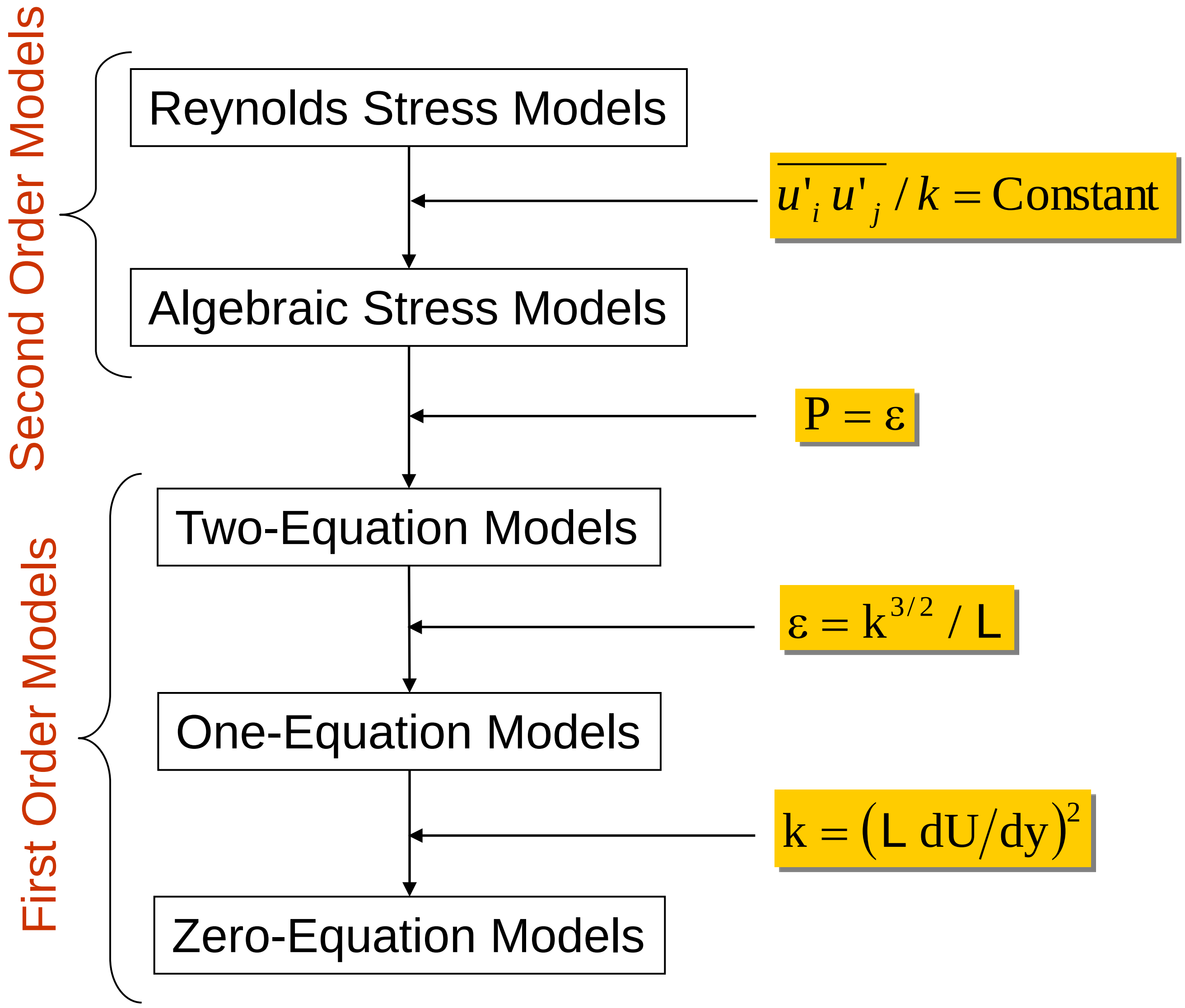
  -) One-equation models: k-model, μ_t-model, etc.

  -) Two-equation models: k-ε, k-kl, k-ω^2, low-Reynolds k-ε, etc.

  -) Algebraic stress models: ASM, etc.

  -) Reynolds stress models: RSM, etc.

THE CLOSURE PROBLEM



First-Order Models

- Governing equations of turbulent stresses and fluxes are very complex ⇒ Analogy between laminar and turbulent flows.

- Laminar Flow:

$$\left\{ \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_j}{\partial x_j} \right.$$

- Turbulent Flow:

$$\left\{ \tau_{ij}^t = -\rho\overline{u_i u_j} = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\rho k \right.$$

Generalized Boussinesq Hypothesis

First-Order Models

- **Zero Equation Models**
  - As the name implies, these models do not make use of any additional equation. In such models we contend with a simple algebraic relation for the turbulent viscosity
  - Zero equation models are based on the mixing length theory which is the length over which there is high interaction between vortices
  - Note that the turbulent viscosity is not a property of the fluid! It is a flow property

Zero-Equation Models

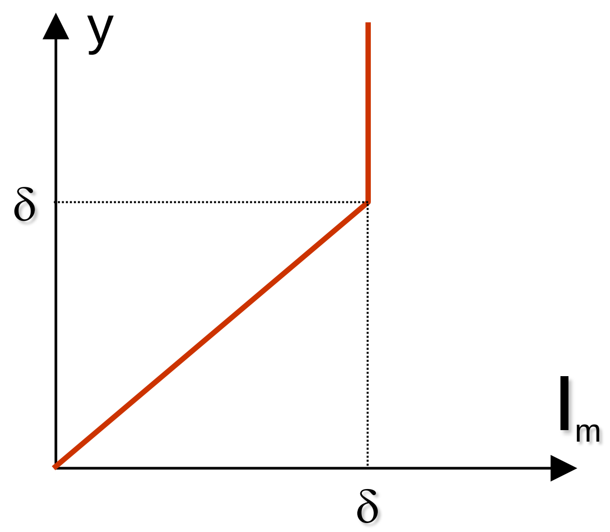
- **Mixing Length Theory**: Dimensional analysis $\Rightarrow$

$$\nu_t = \frac{\mu_t}{\rho} \propto lu = l_m \left(l_m \frac{dU}{dy} \right)$$

l_m is determined experimentally. For boundary layers :

$l_m = \kappa y$ for $y<\delta$

$l_m = \delta$ for $y \geq \delta$



First-Order Models

- **One Equation Models**
  - As the name implies, we develop one equation for a given variable. The variable of choice is the turbulent kinetic energy
  - We write a transport equation for turbulent kinetic energy and use algebraic modeling for the new variables introduced

One-Equation Models

- **Turbulent Kinetic Energy:** $k = \frac{1}{2}\left(\overline{u^2} + \overline{v^2} + \overline{w^2}\right)$

  Dimensional analysis $\Rightarrow \mu_t \propto \mathrm{u}\,\mathrm{l}$

  u : velocity scale - proportional to $k^{1/2}$
  l : length scale – mixing length l_m

  $$\mu_t = C_\mu \sqrt{k}\, l_m$$

- Other choices: $\mathrm{u} \propto \overline{uv}^{1/2}$ (Bradshaw et al.), etc.
- Direct calculation of μ_t using a transport equation for the eddy viscosity (Nee & Kovasznay).

One-Equation Models

- **Turbulent Kinetic Energy:**

$$\frac{\partial \rho k}{\partial \tau} + \frac{\partial \rho U_j k}{\partial x_j} = \boxed{-\rho \overline{u_i u_j} \frac{\partial U_i}{\partial x_j}} - \boxed{\beta g_i \rho \overline{u_i t}} \cdots (P_k + G_k)$$

$$- \frac{\partial}{\partial x_j}\left(\frac{1}{2}\boxed{\rho \overline{u_i^2 u_j} + \overline{p u_j}} - \mu \frac{\partial k}{\partial x_j} \right)(D_k)$$

$$- \mu \boxed{\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}} \cdots\cdots\cdots\cdots (\varepsilon_k)$$

- Unknown Correlations that require modeling:
  -) Turbulent & Pressure Diffusion;
  -) Dissipation Rates.

One-Equation Models

- **Turbulent Kinetic Energy Equation:**

$$\frac{\partial \rho k}{\partial \tau} + \frac{\partial \rho U_j k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + P_k + G_k - \rho\varepsilon$$

$$P_k = \mu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} \quad ; \quad G_k = \beta g_i \frac{k_t}{c_p}\frac{\partial T}{\partial x_i}$$

Dimensional analysis (Harlow & Nakayama) $\Rightarrow$

$$\varepsilon \propto k^{3/2}/l = C_d\, k^{3/2}/l_m$$

Conclusion

- **Turbulence is a very important phenomenon**
- **Turbulence modeling is a highly active area of research**
- **The future of CFD has a high dependence on turbulence models**
- **LES seems to be the prevailing method for the next two decades**
- **DNS, with the ever increasing computational power and parallel computing technologies, remains a dream for all researchers. Will it ever be feasible on a personal computer?**