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Extension of Kelvin's Minimum Energy Theorem to Flows with Open Regions

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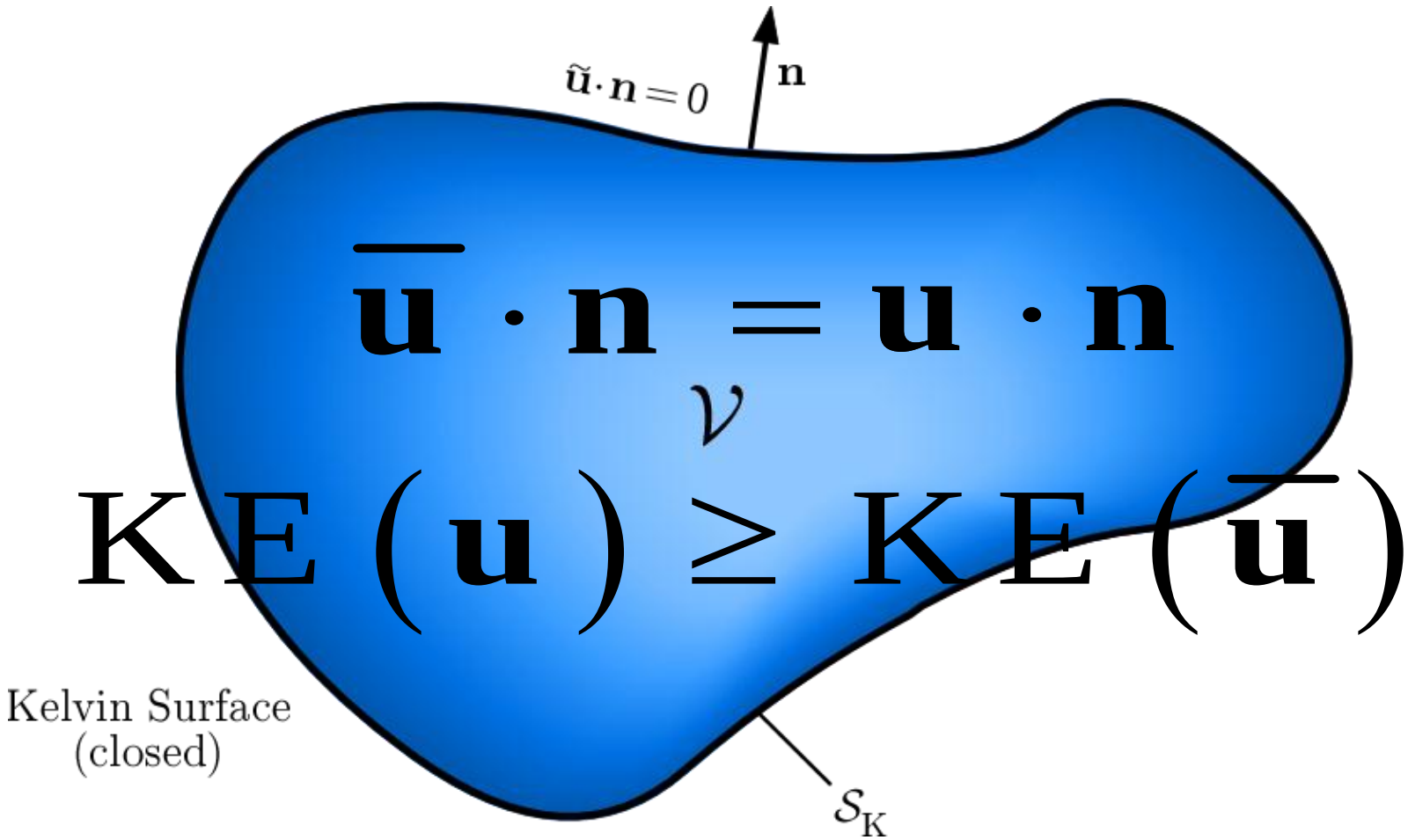
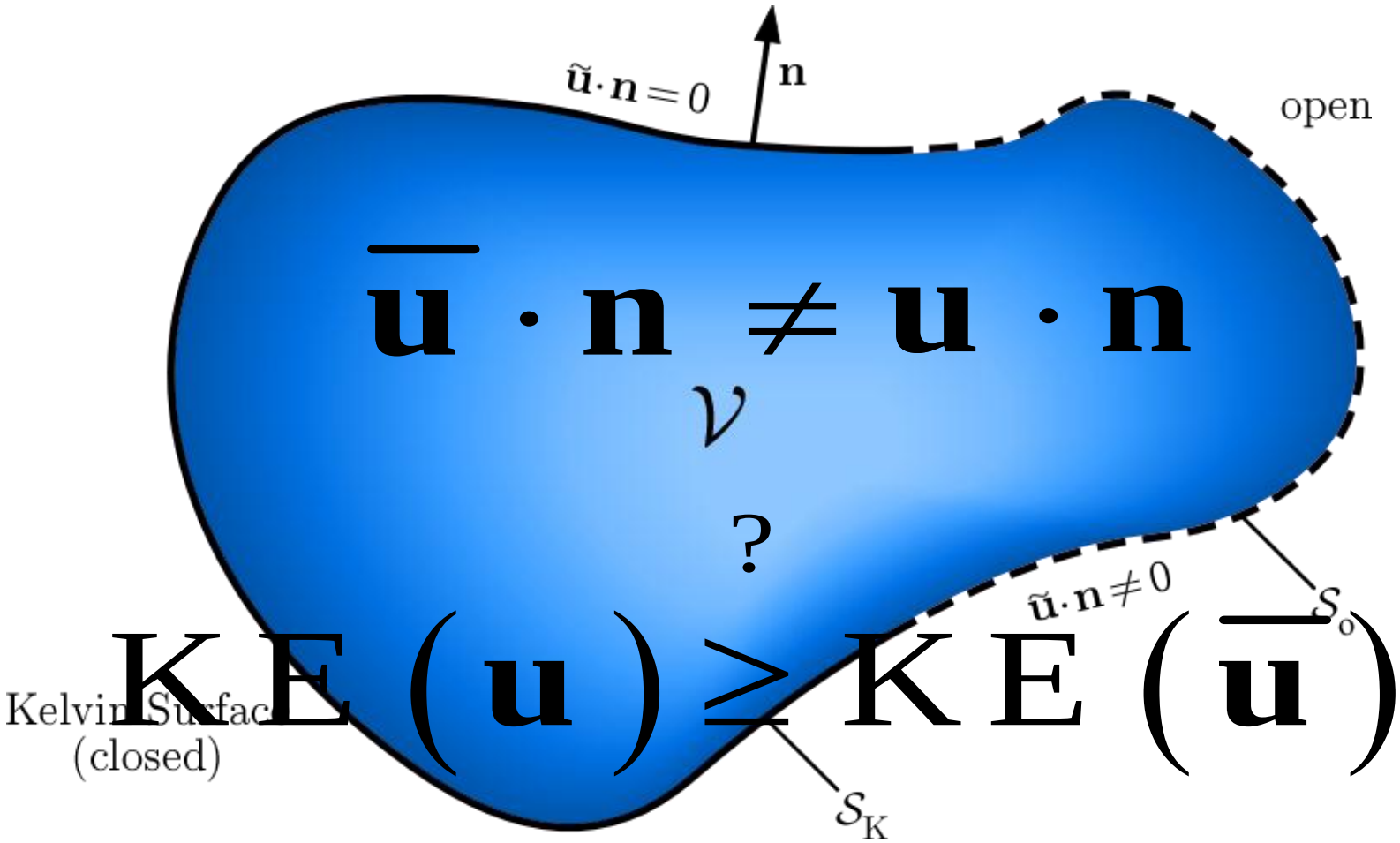
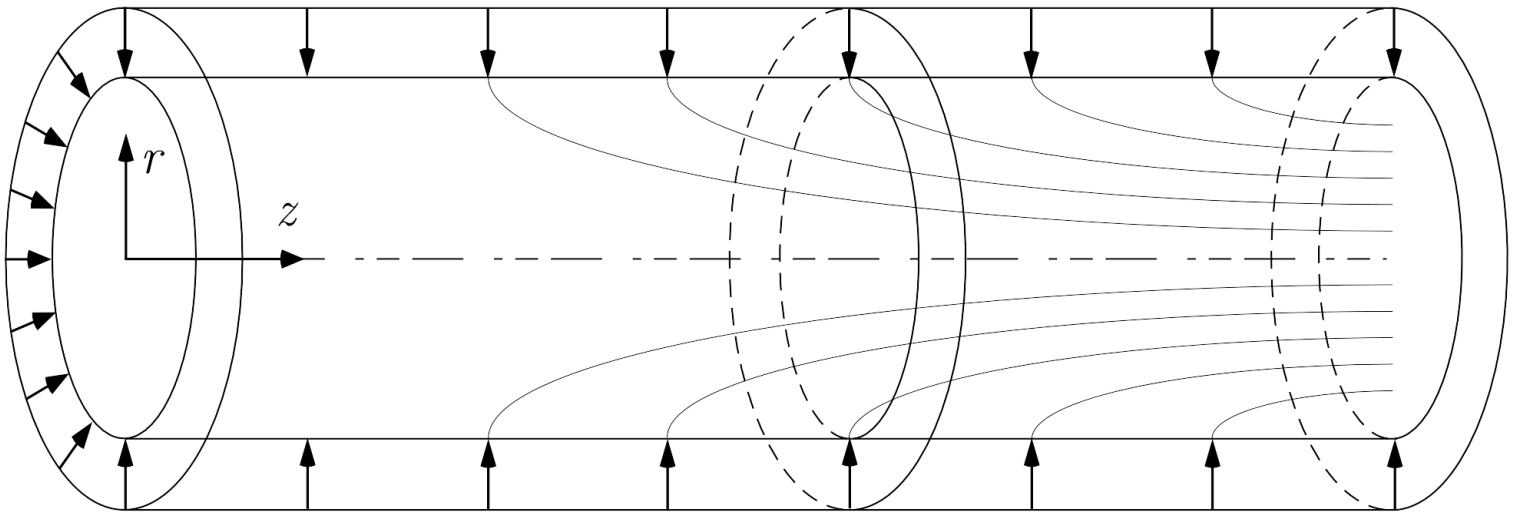




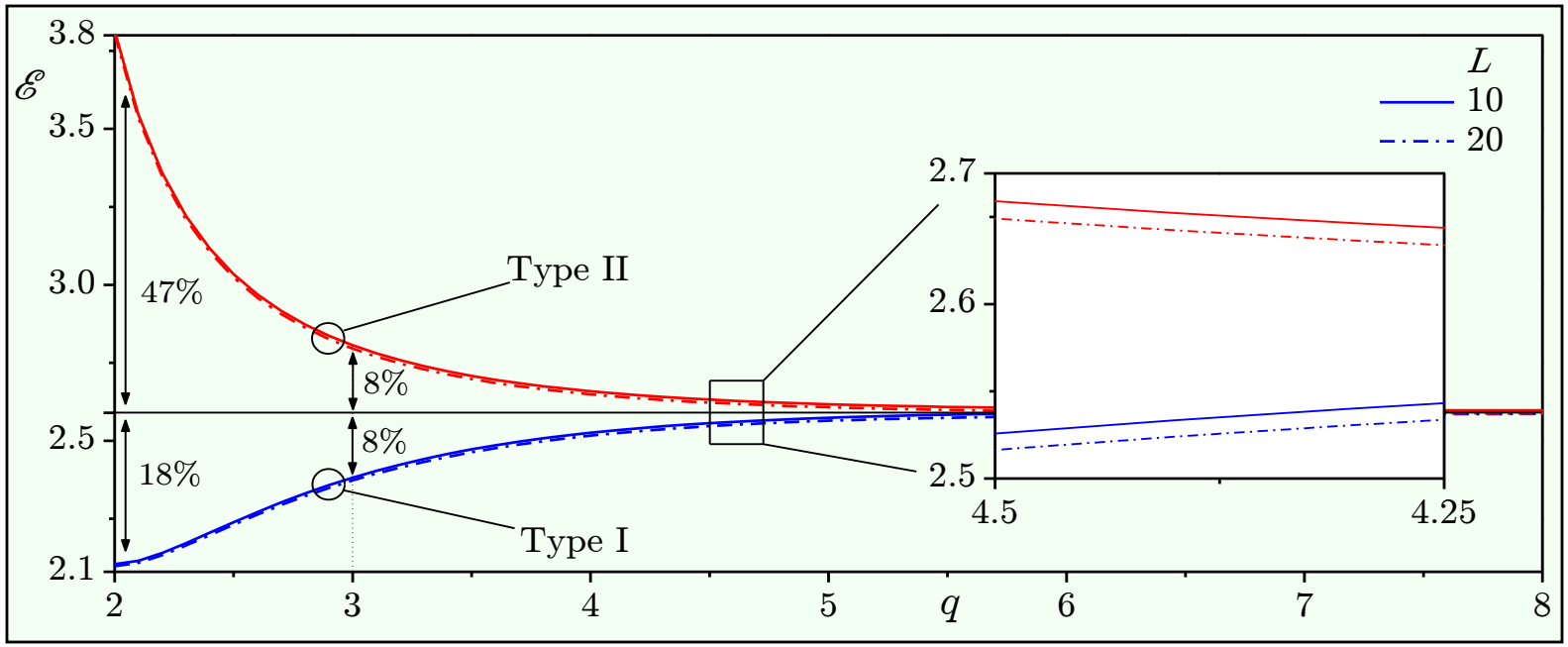


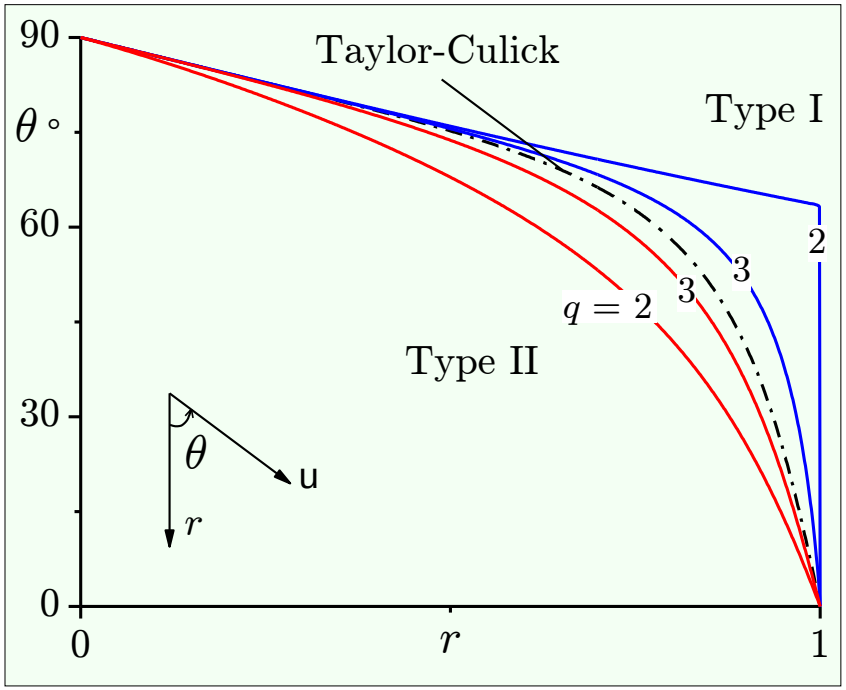
- Saad, T., and Majdalani, J., "On the Lagrangian Optimization of Wall-Injected Flows: from the Hart-McClure Potential to the Taylor-Culick Rotational Motion," Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. (10.1098/rspa.2009.0326)
- Saad, T., and Majdalani, J., "Energy Based Solutions of the Bidirectional Vortex with Multiple Mantles," 45th AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit, Denver, Colorado, USA, August 2-5, 2009.
- Saad, T., and Majdalani, J., "Energy Based Mean Flow Solutions for Slab Hybrid Rocket Chambers," 44th AIAA/ASME/SAE/ASEE Joint Propulsion Conference & Exhibit, Hartford, Connecticut, USA, July 20-23, 2008.
- Majdalani, J., and Saad, T., "Energy Steepened States of the Taylor-Culick Profile," AIAA Paper 2007-5797, July 2007.

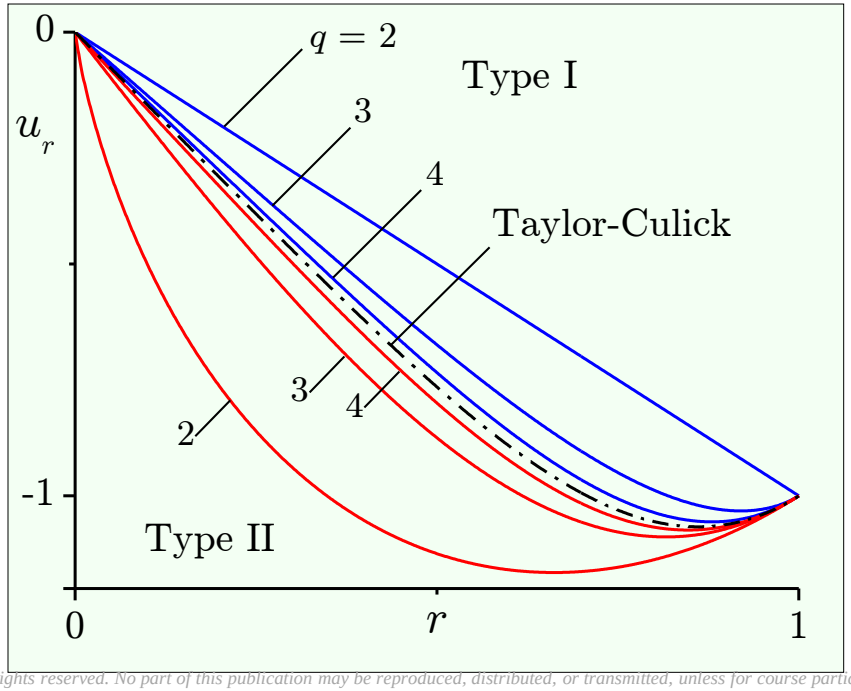
$$\psi(r,z) = \sum_{n=0}^{\infty} \alpha_n z \sin\left[\tfrac{1}{2}(2n+1)\pi r^2\right]$$

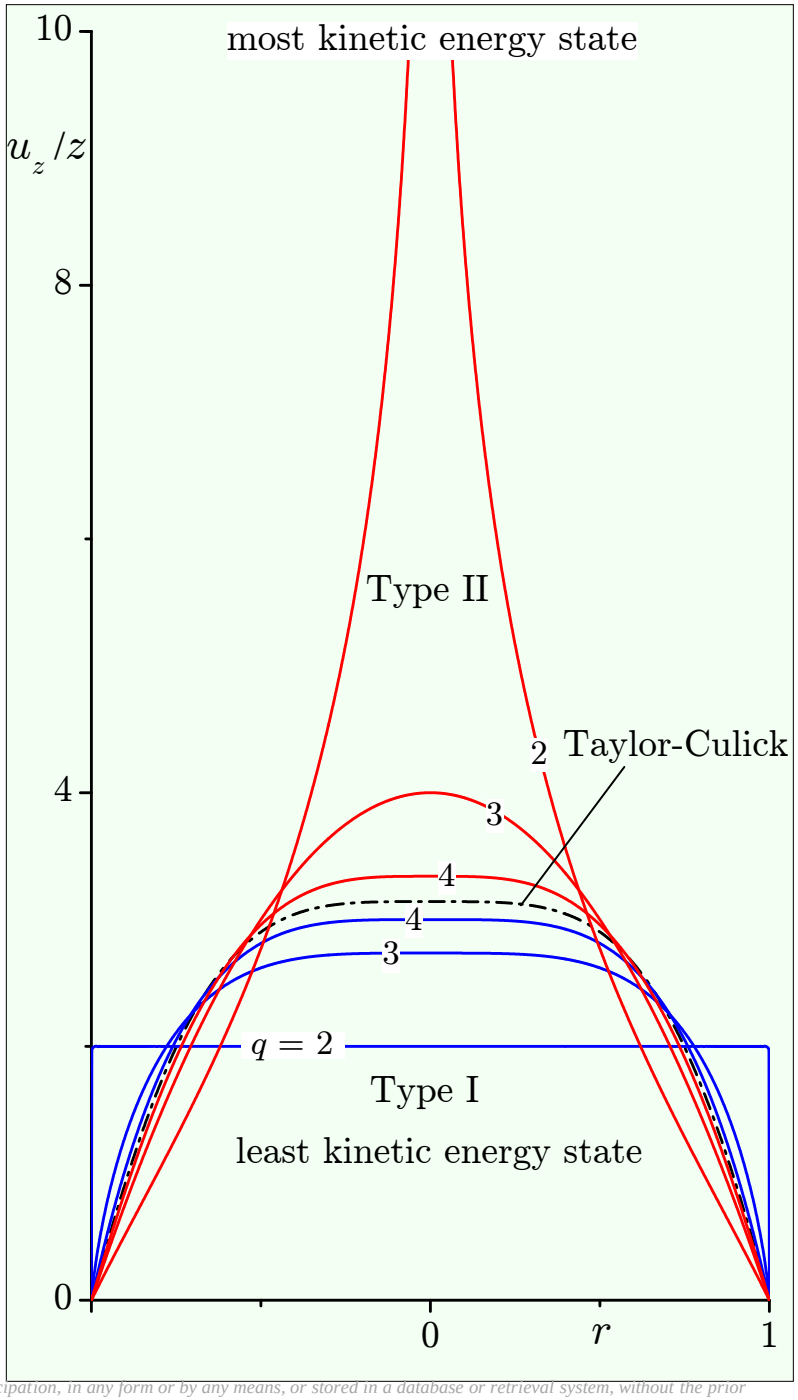
$$\alpha_n^-(q) = \frac{(-1)^n (2n+1)^{-q}}{\sum_{k=0}^{\infty}(2k+1)^{-q}} = \frac{(-1)^n (2n+1)^{-q}}{\zeta(q)(1-2^{-q})}$$

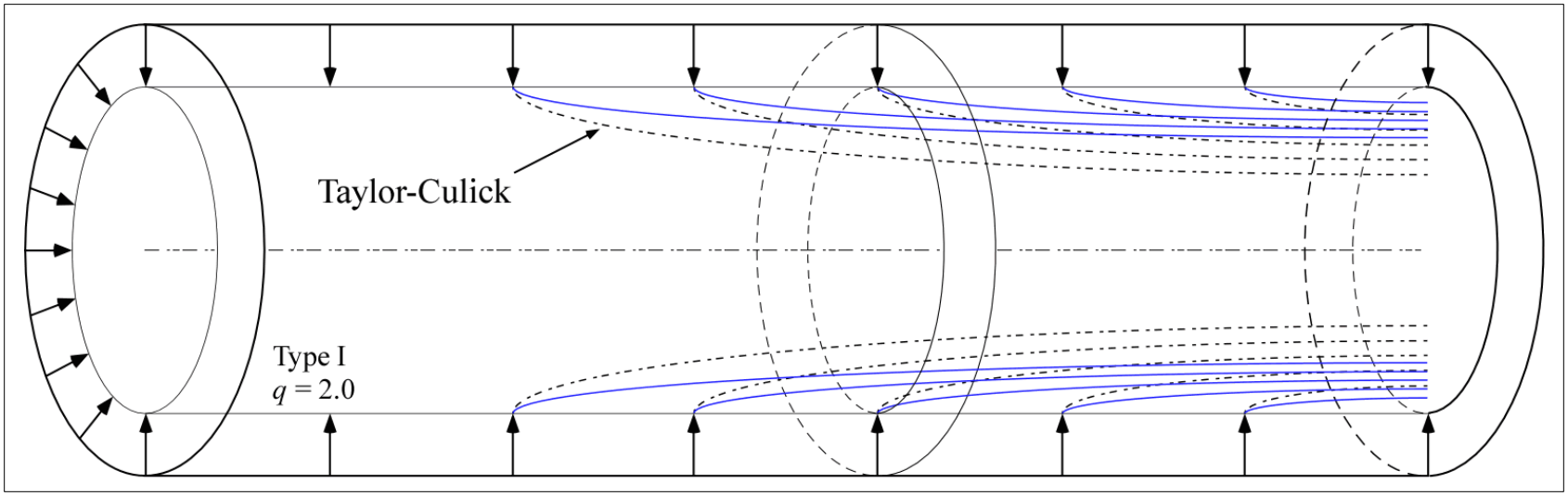
$$\alpha_n^+(q) = \frac{(2n+1)^{-q}}{\sum_{k=0}^{\infty}(-1)^k (2k+1)^{-q}} = \frac{4^q (2n+1)^{-q}}{\zeta(q,\tfrac{1}{4}) - \zeta(q,\tfrac{3}{4})}$$

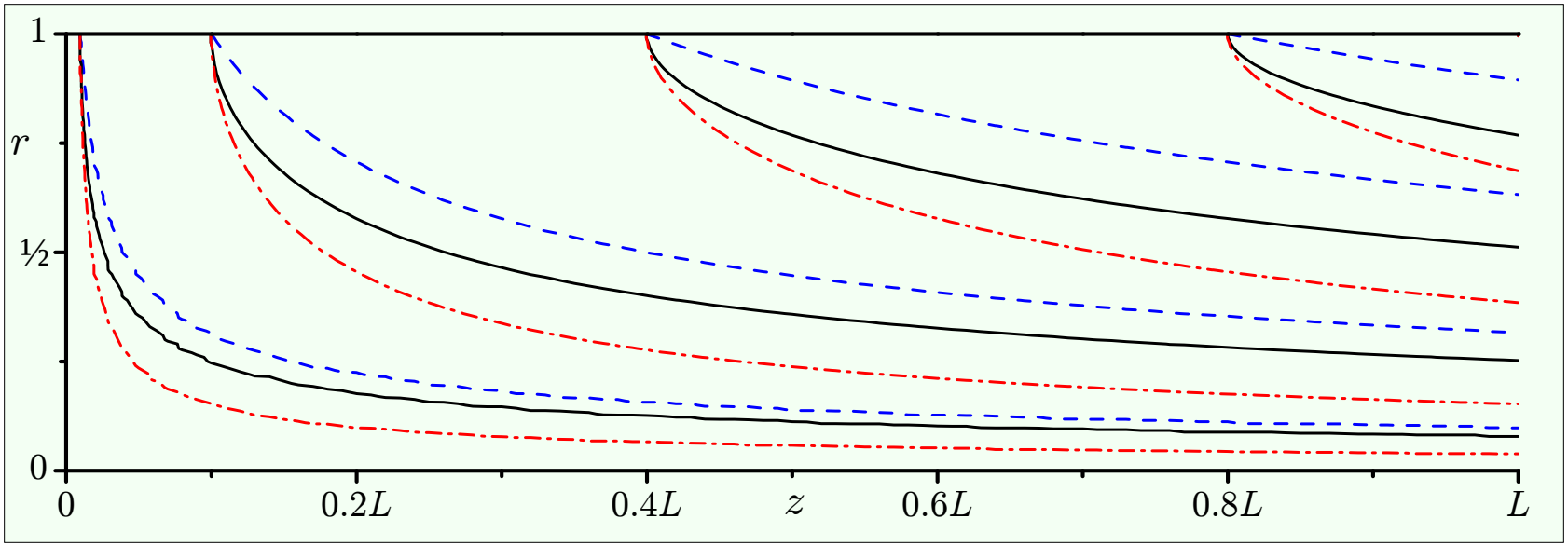




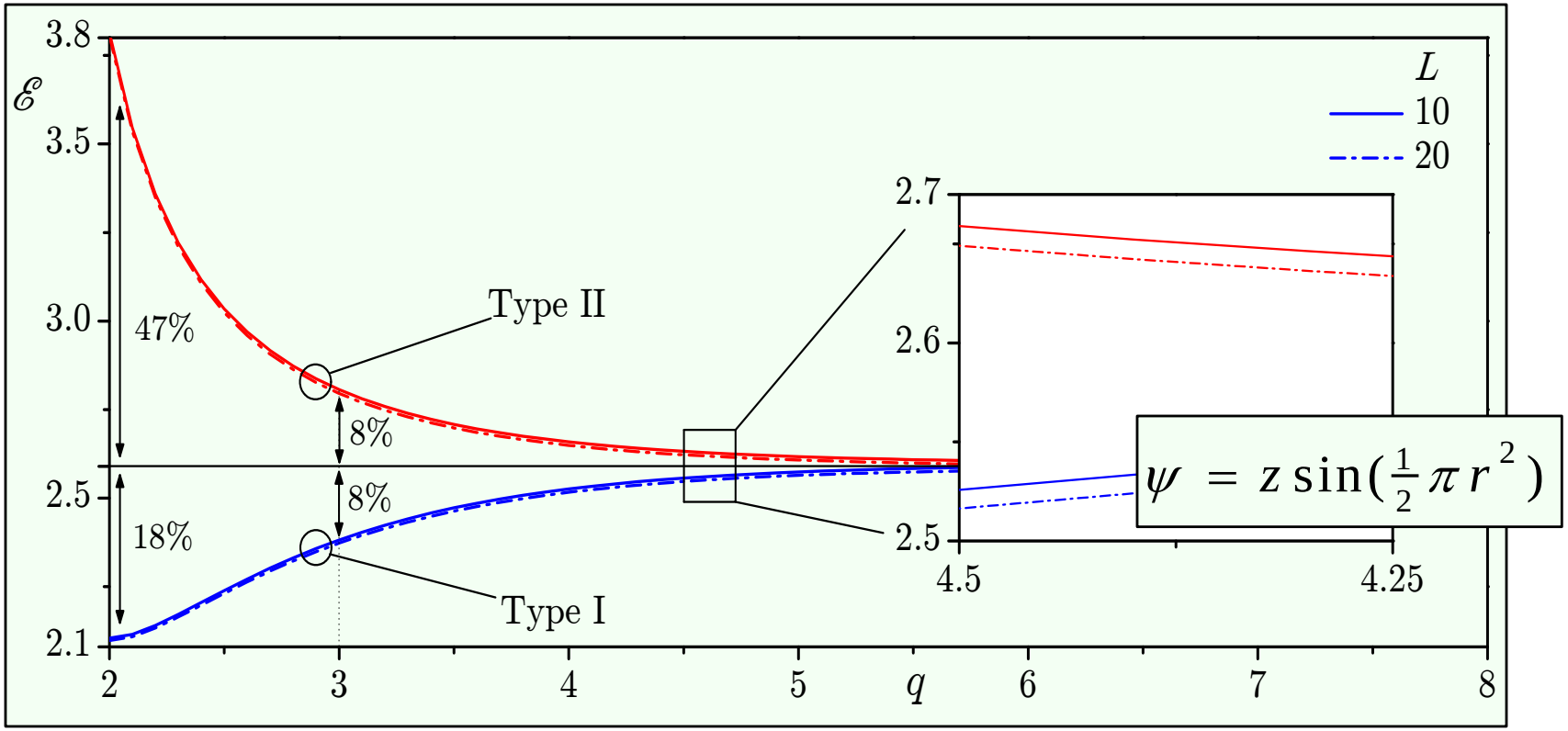




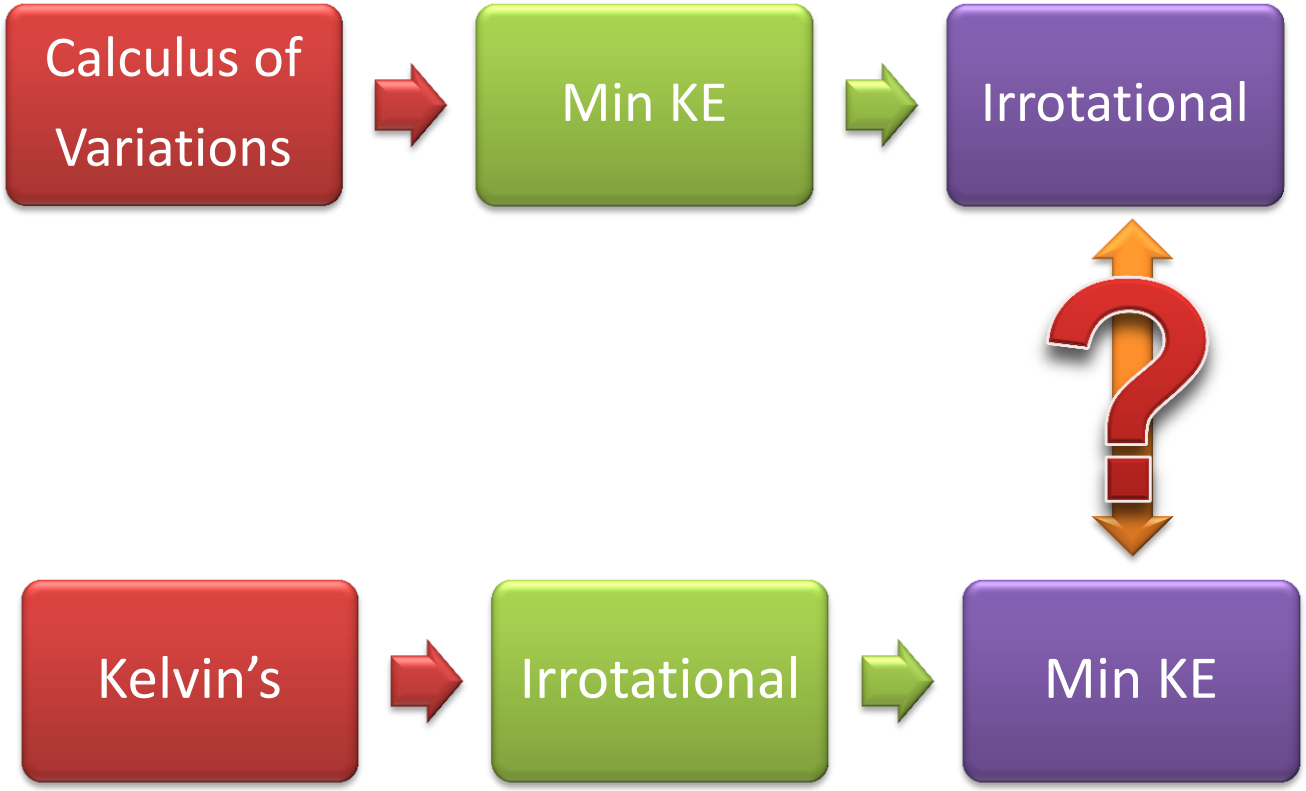




$$\psi = \frac{1}{C} z \sum_{n=0}^{\infty} \frac{\sin \eta}{(2n+1)^2}$$

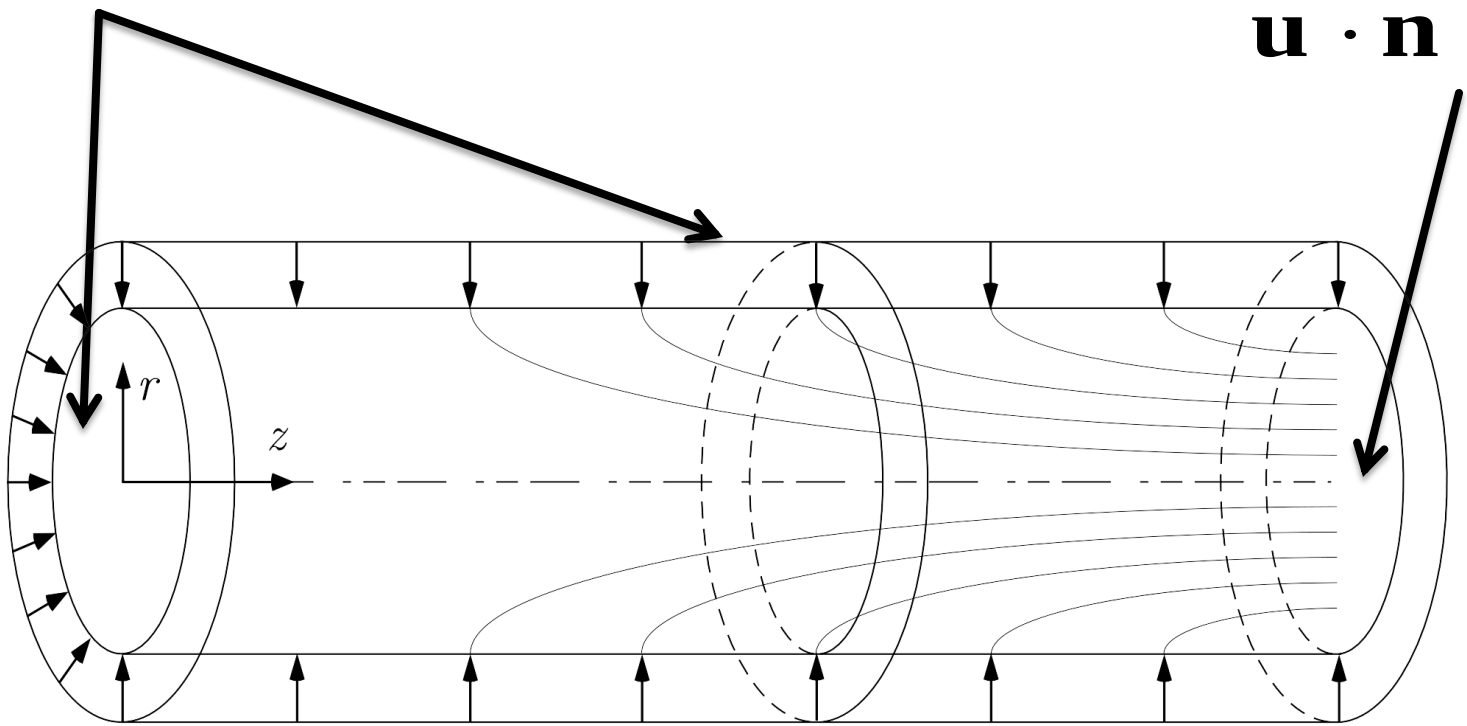


$$\psi = \frac{8}{\pi^2} z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \eta \mapsto r^2 z$$

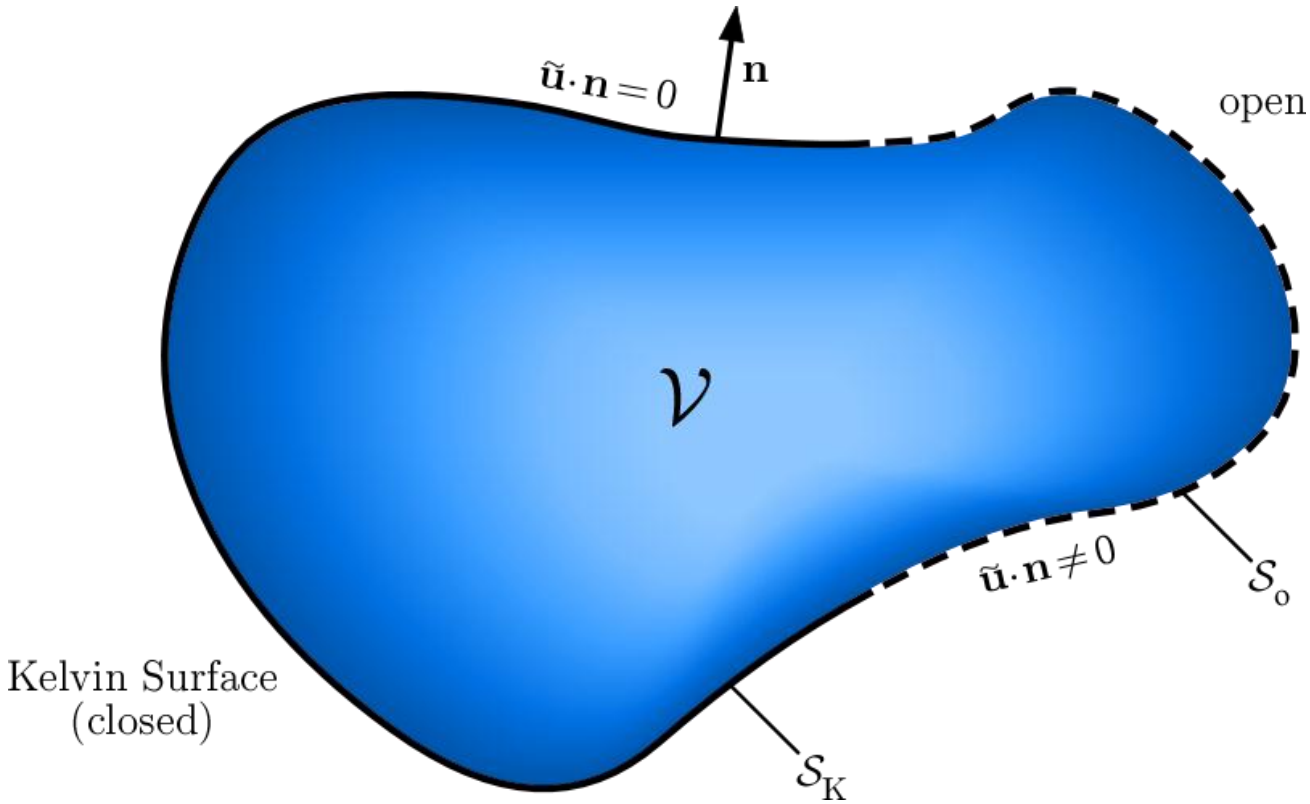


$$\tilde{\mathbf{u}} \cdot \mathbf{n} = 0$$

$$\tilde{\mathbf{u}} \cdot \mathbf{n} \neq 0$$

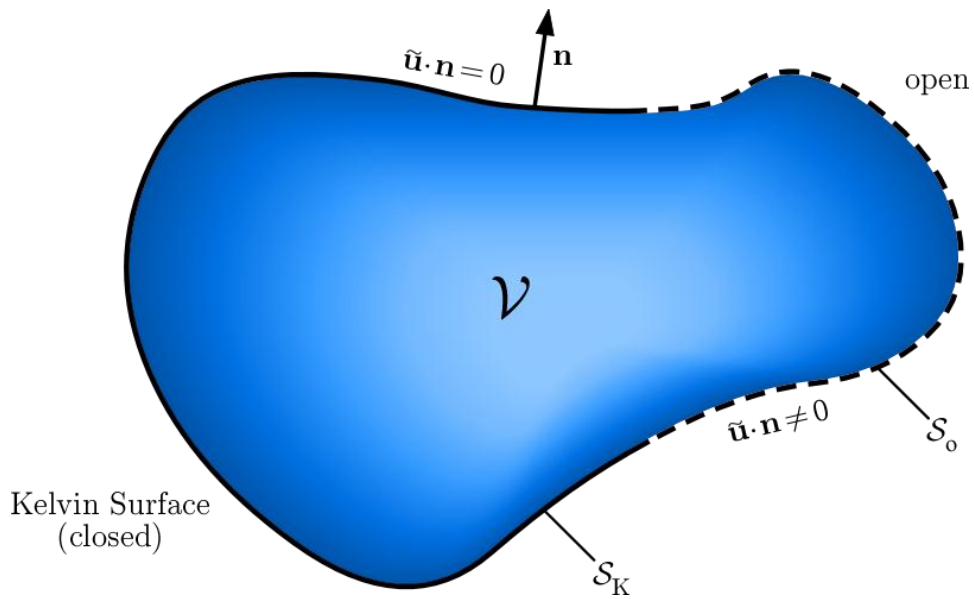


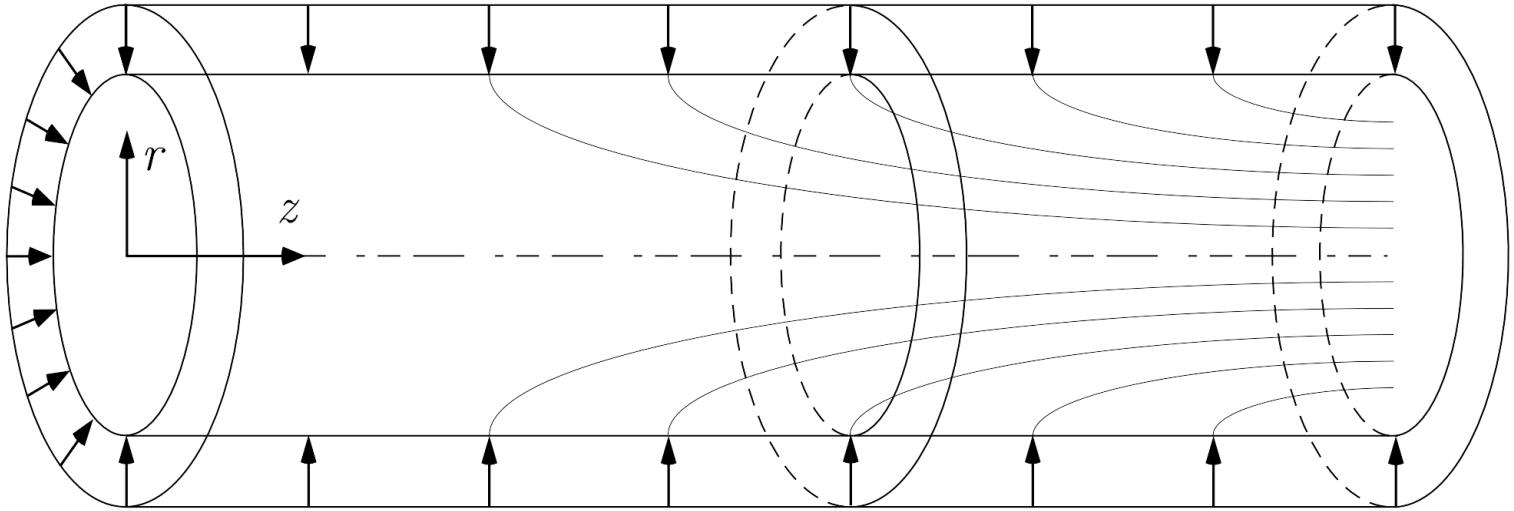
- Extend Kelvin's theorem to regions with "open boundaries"



$$\begin{cases} \mathcal{S}_{\mathrm{K}}: & \tilde{\mathbf{u}}\cdot\mathbf{n} = \left(\mathbf{u} - \nabla\phi\right)\cdot\mathbf{n} = 0 \\ \mathcal{S}_{\mathrm{o}}: & \tilde{\mathbf{u}}\cdot\mathbf{n} \neq \mathbf{u}\cdot\mathbf{n} \neq 0. \end{cases}$$

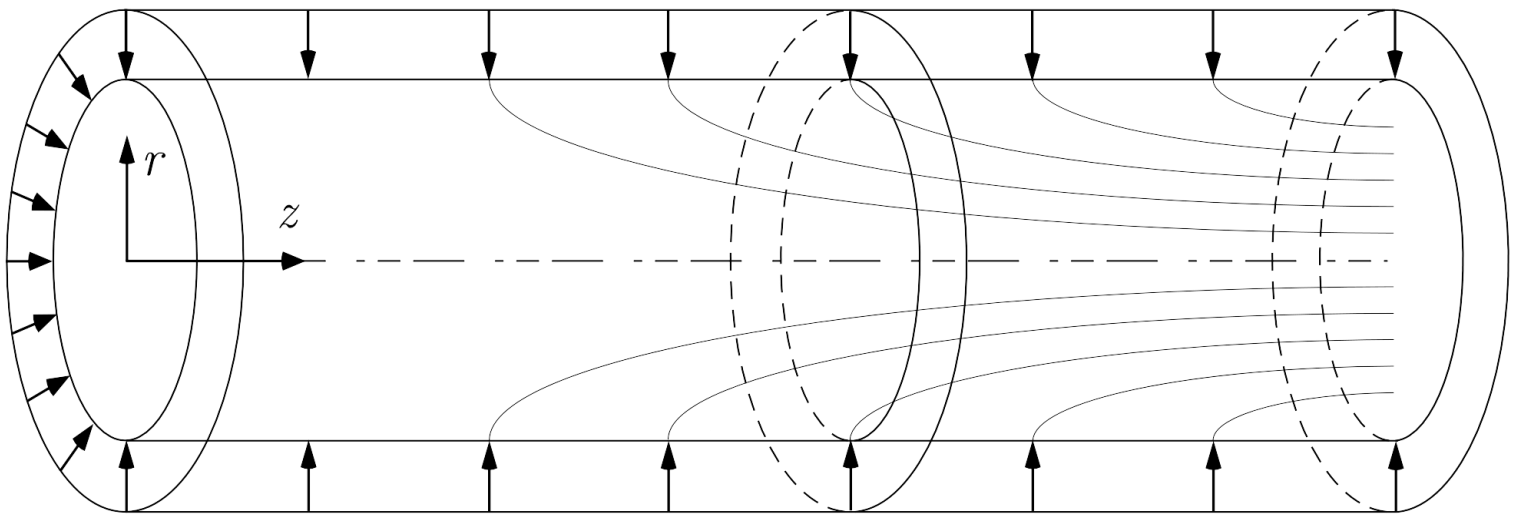
$$\begin{cases} T_{\mathrm{o}} = \iint\limits_{\mathcal{S}_{\mathrm{o}}} \phi \tilde{\mathbf{u}} \cdot \mathbf{n} \, \mathrm{d}\mathcal{S} \geq 0 & \text{(sufficient condition)} \\ \text{else (if } T_{\mathrm{o}} < 0) \\ T_{\mathrm{o}} > -\frac{1}{2} \iiint\limits_{\mathcal{V}} \tilde{\mathbf{u}}^{2} \, \mathrm{d}\mathcal{V} & \text{(necessary + sufficient condition)} \end{cases}$$





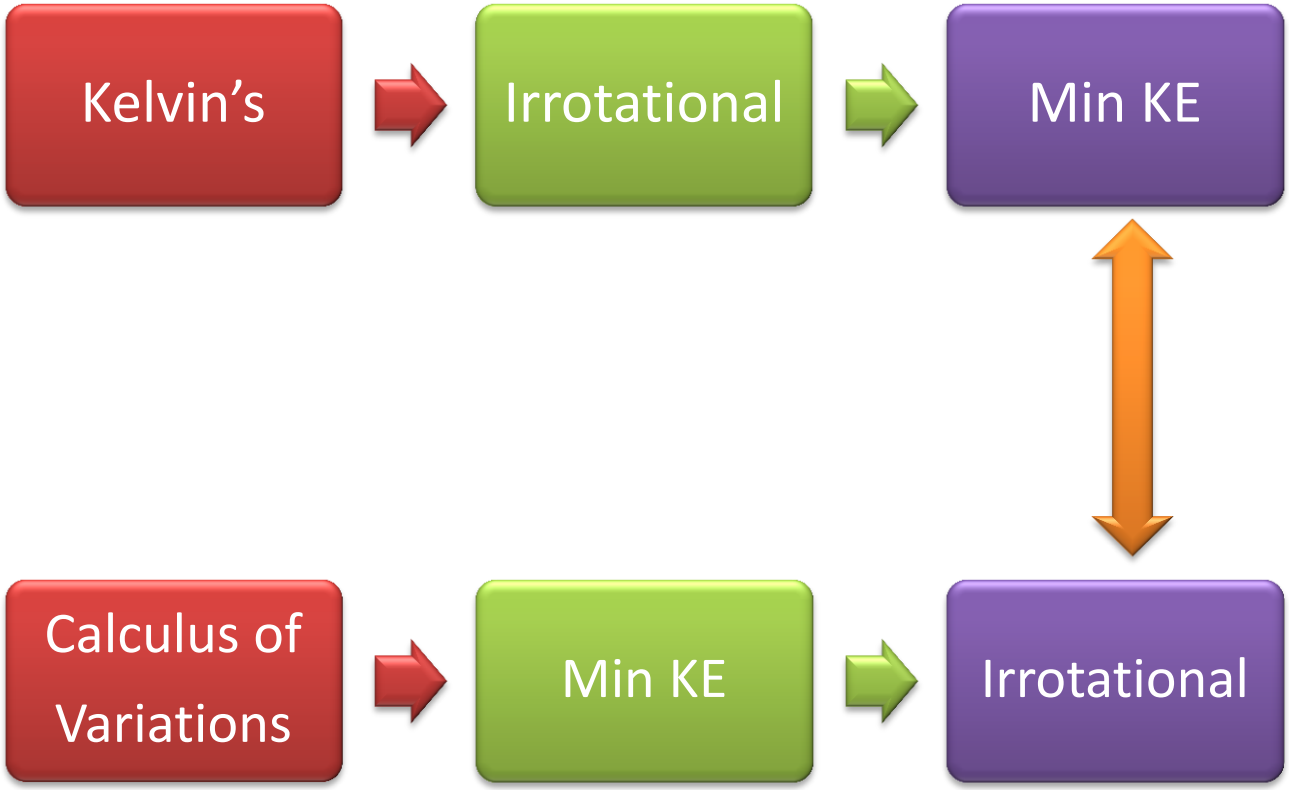
$$\phi = -\frac{1}{2}r^2 + z^2 \qquad \bar{\mathbf{u}} = -r\mathbf{e}_r + 2z\mathbf{e}_z$$

$$\tilde{\mathbf{u}} = \mathbf{u} - \bar{\mathbf{u}} = \left(-r^{-1}\sum \alpha_n \sin\eta + r\right)\mathbf{e}_r + \left[\pi \sum \alpha_n z(2n+1)\cos\eta - 2z\right]\mathbf{e}_z$$



$$T_o = \iint_{\mathcal{S}_o} \phi \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} \, \mathrm{d}\mathcal{S} = \left[2 \sum \alpha_n \left(2n+1 \right)^{-1} - \tfrac{1}{2}\pi \right] L \geq 0$$

$$2 \sum \alpha_n \left(2n+1 \right)^{-1} \geq \tfrac{1}{2}\pi ; \quad \forall \; q \geq 2$$

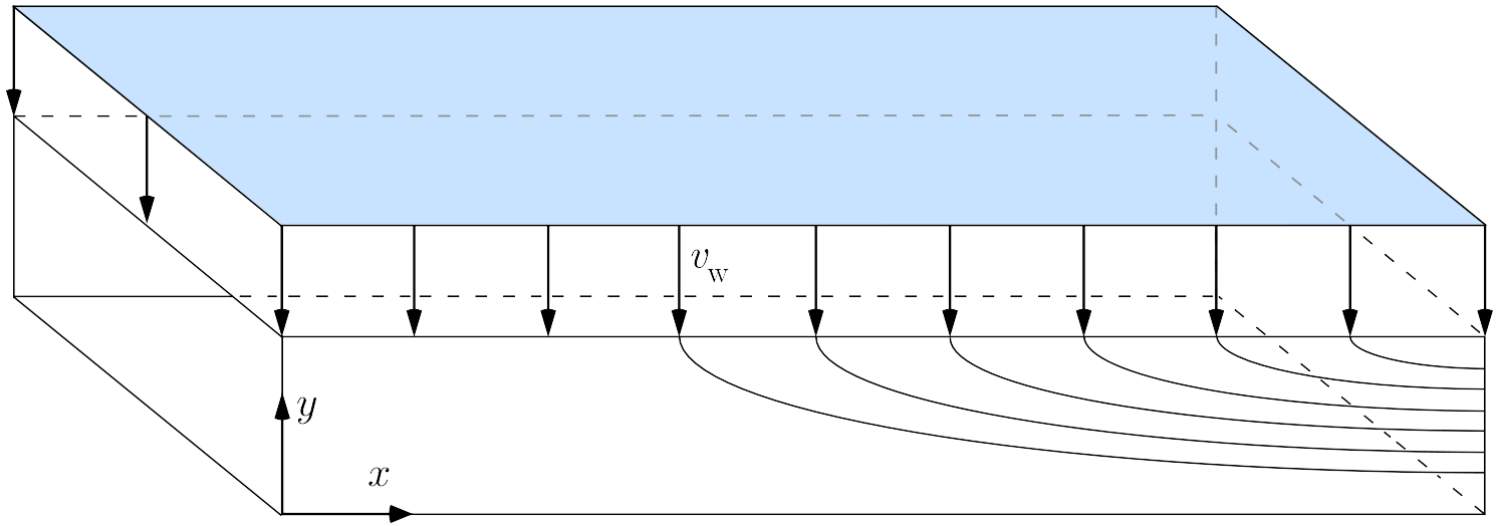


Mass Equiflux Principle

- In a simply connected fluid region V, and for a well posed set of boundary conditions, the mass flux of the irrotational motion is equal to that of any other rotational motion that may be established under the same conditions
- Can be used to determine unique potential function

$$\int_{S_o} \overline{\mathbf{u}} \cdot \mathbf{n} \, \mathrm{d}S = \int_{S_o} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S$$

Porous Channel



$$\mathbf{u} = \tfrac{1}{2}\pi x \cos\left(\tfrac{1}{2}\pi y\right)\mathbf{i} - \sin\left(\tfrac{1}{2}\pi y\right)\mathbf{j}$$

$$\phi = \tfrac{1}{2}\left(x^2 - y^2\right) \qquad \overline{\mathbf{u}} = x\mathbf{i} - y\mathbf{j}$$

$$T_o = \iint_{\mathcal{S}_o} \phi\tilde{\mathbf{u}} \cdot \mathbf{n}\,\mathrm{d}\mathcal{S} = \left(4\pi^{-2} - \tfrac{1}{3}\right)L > 0$$

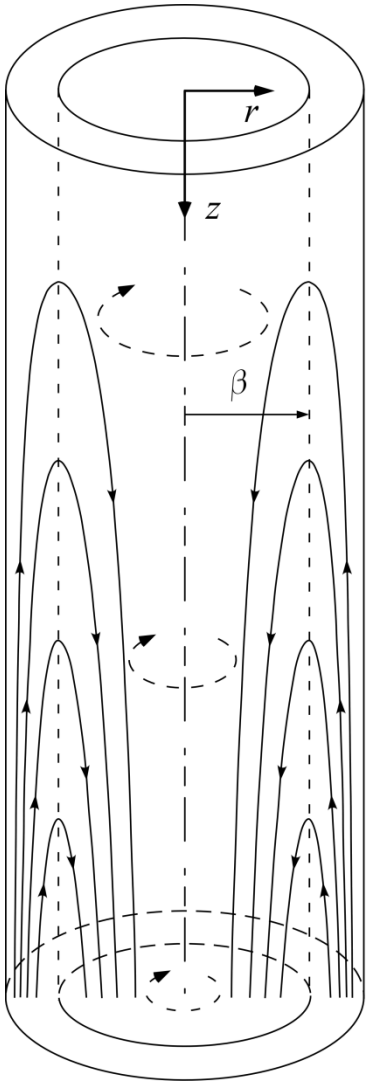
Bidirectional Vortex - Lamellar

$$\mathbf{u} = -r^{-1}\sin\left(\pi r^2\right)\mathbf{e}_r + r^{-1}\mathbf{e}_\theta + 2\pi z\cos\left(\pi r^2\right)\mathbf{e}_z$$

$$\phi = \begin{cases} -\frac{1}{2}a_0 r^2 + \theta + a_0 z^2; & 0 \le r < \beta \\ -b_0\left(\frac{1}{2}r^2 - \ln r\right) + \theta + b_0 z^2; & \beta < r \le 1 \end{cases}$$

$$a_0 = -\beta^{-1}u_r(\beta) \qquad b_0 = \frac{\beta^2 a_0}{\beta^2 - 1}$$

$$T_o = \iint_{\mathcal{S}_o} \phi\tilde{\mathbf{u}}\cdot\mathbf{n}\,\mathrm{d}\mathcal{S} \approx 0.596L > 0$$

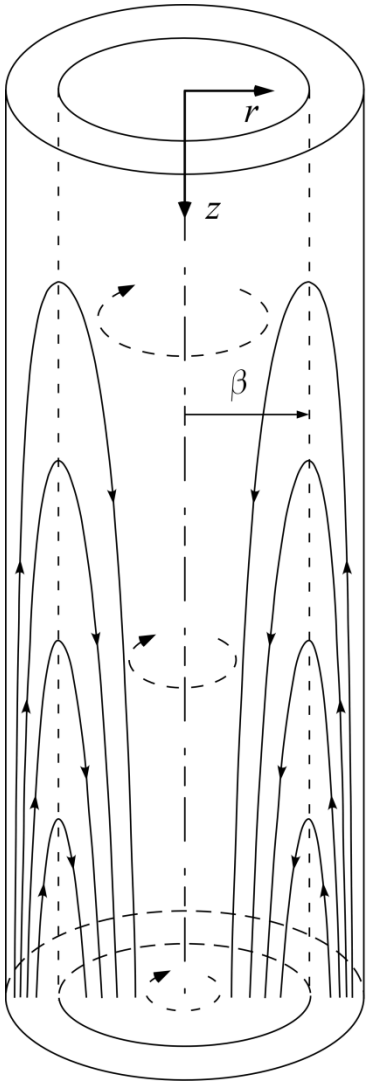


Bidirectional Vortex - Beltramian

$$\psi = \begin{cases} crzJ_1(\lambda_0 r); & \text{type I} \\ cLr\sin\left(\frac{1}{2}\pi zL^{-1}\right)J_1(\lambda_0 r); & \text{type II} \end{cases}$$

$$\phi = \begin{cases} -\frac{1}{2}a_0 r^2 + \theta + a_0 z^2; & 0 \le r < \beta \\ -b_0\left(\frac{1}{2}r^2 - \ln r\right) + \theta + b_0 z^2; & \beta < r \le 1 \end{cases}$$

$$T_o = \iint_{\mathcal{S}_o} \phi\tilde{\mathbf{u}}\cdot\mathbf{n}\,\mathrm{d}\mathcal{S} \approx 5.163\times10^{-3}L^3 + 0.8L \ge 0$$



Conclusions

- Extended Kelvin's theorem to regions with "open boundaries"
- Kelvin's theorem holds as long as one of two simple conditions is satisfied
- Identified a new surface decomposition (Kelvin and open boundaries)
- Mass equiflux principle: determine unique potentials
- For flows that were considered, the sufficient condition was always satisfied
- Extension to compressible flows is in preparation