Extension of Kelvin’s Minimum Energy Theorem to Flows with Open Regions

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Extension of Kelvin’s Theorem to Open Regions

\( \mathbf{\bar{u}} \cdot \mathbf{n} = u \cdot \mathbf{n} \)

\( \nu \)

\( \text{KE} \left( \mathbf{u} \right) \geq \text{KE} \left( \mathbf{\bar{u}} \right) \)

Kelvin Surface
(closed)
Extension of Kelvin's Theorem to Open Regions

$$\mathbf{u} \cdot \mathbf{n} \neq \mathbf{u} \cdot \mathbf{n}$$

$$\bar{\mathbf{u}} \cdot \mathbf{n} = 0$$

$$\mathbf{KE} (\mathbf{u}) \geq \mathbf{KE} (\overline{\mathbf{u}})$$

Kelvin Surface (closed)

$S_K$

$S_0$

open


\[ \psi (r, z) = \sum_{n=0}^{\infty} \alpha_n z \sin \left[ \frac{1}{2} (2n + 1) \pi r^2 \right] \]

\[ \alpha_n^-(q) = \frac{(-1)^n (2n+1)^{-q}}{\sum_{k=0}^{\infty} (2k+1)^{-q}} = \frac{(-1)^n (2n+1)^{-q}}{\zeta(q)(1-2^{-q})} \]

\[ \alpha_n^+(q) = \frac{(2n+1)^{-q}}{\sum_{k=0}^{\infty} (-1)^k (2k+1)^{-q}} = \frac{4^q (2n+1)^{-q}}{\zeta(q, \frac{1}{2}) - \zeta(q, \frac{3}{4})} \]
Extension of Kelvin’s Theorem to Open Regions

Taylor-Culick

Type I

$u_r$ / $z$

$u_z / z$

$q = 2$

Type II

$\theta$

$r$

most kinetic energy state

least kinetic energy state

$
\mu_r$

Taylor-Culick

Type I

$\mu$

$
\mu°$

Type II

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Extension of Kelvin’s Theorem to Open Regions

Taylor-Culick

Type I

\( q = 2.0 \)

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\[ \psi \left( \phi, \eta \right) = \frac{1}{C} z \sum_{n=0}^{\infty} \frac{\sin \eta}{(2n + 1)^2} \]

\[ \psi = z \sin \left( \frac{1}{2} \pi r^2 \right) \]

\[ \psi = z \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^2} \frac{\sin \eta}{(2n + 1)^2} \rightarrow r^2 z \]
Extension of Kelvin’s Theorem to Open Regions

Calculus of Variations → Min KE → Irrotational

Kelvin’s → Irrotational → Min KE

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\[ \tilde{u} \cdot n = 0 \]

\[ \tilde{u} \cdot n \neq 0 \]
Extend Kelvin’s theorem to regions with “open boundaries”

\[
\begin{align*}
\tilde{u} \cdot n &= 0 \\
\n K_0 : 0 &\quad \phi \big|_S = 0 \\
\n S_K : \tilde{u} \cdot n = (u - \nabla \phi) \cdot n &= 0 \\
S_o : \tilde{u} \cdot n \neq u \cdot n \neq 0.
\end{align*}
\]
\[
\begin{cases}
T_o = \int_\mathcal{S}_o \phi \tilde{u} \cdot n \, dS \geq 0 & \text{(sufficient condition)} \\
\text{else (if } T_o < 0) \\
T_o > -\frac{1}{2} \int_\mathcal{V} \int_\mathcal{V} \tilde{u}^2 \, dV & \text{(necessary + sufficient condition)}
\end{cases}
\]
\[
\phi = -\frac{1}{2} r^2 + z^2 \quad \overline{u} = -r e_r + 2z e_z
\]

\[
\tilde{u} = u - \overline{u} = \left( -r^{-1} \sum \alpha_n \sin \eta + r \right) e_r + \left[ \pi \sum \alpha_n z (2n + 1) \cos \eta - 2z \right] e_z
\]
Extension of Kelvin’s Theorem to Open Regions

\[ T_o = \iint_{S_o} \phi \tilde{u} \cdot n \, dS = \left[ 2 \sum \alpha_n (2n + 1)^{-1} - \frac{1}{2} \pi \right] L \geq 0 \]

\[ 2 \sum \alpha_n (2n + 1)^{-1} \geq \frac{1}{2} \pi ; \quad \forall \ q \geq 2 \]
Extension of Kelvin’s Theorem to Open Regions

Kelvin’s → Irrotational → Min KE

Calculus of Variations → Min KE → Irrotational
Mass Equiflux Principle

In a simply connected fluid region $V$, and for a well posed set of boundary conditions, the mass flux of the irrotational motion is equal to that of any other rotational motion that may be established under the same conditions.

Can be used to determine unique potential function

\[ \int_{S_0} \bar{\mathbf{u}} \cdot \mathbf{n} \, dS = \int_{S_0} \mathbf{u} \cdot \mathbf{n} \, dS \]
Porous Channel

\[ u = \frac{1}{2} \pi x \cos \left( \frac{1}{2} \pi y \right) \mathbf{i} - \sin \left( \frac{1}{2} \pi y \right) \mathbf{j} \]

\[ \phi = \frac{1}{2} \left( x^2 - y^2 \right) \]

\[ \bar{u} = x \mathbf{i} - y \mathbf{j} \]

\[ T_o = \iint_{S_o} \phi \bar{u} \cdot \mathbf{n} \, dS = \left( 4\pi^{-2} - \frac{1}{3} \right) L > 0 \]
Bidirectional Vortex - Lamellar

\[ u = -r^{-1} \sin \left( \pi r^2 \right) e_r + r^{-1} e_\theta + 2\pi z \cos \left( \pi r^2 \right) e_z \]

\[ \phi = \begin{cases} 
-\frac{1}{2} a_0 r^2 + \theta + a_0 z^2; & 0 \leq r < \beta \\
-b_0 \left( \frac{1}{2} r^2 - \ln r \right) + \theta + b_0 z^2; & \beta < r \leq 1 
\end{cases} \]

\[ a_0 = -\beta^{-1} u_r (\beta) \quad b_0 = \frac{\beta^2 a_0}{\beta^2 - 1} \]

\[ T_o = \iint_{S_o} \phi \tilde{u} \cdot n \, dS \approx 0.596L > 0 \]
Bidirectional Vortex - Beltramian

\[ \psi = \begin{cases} \text{type I} \\ c rz J_1 (\lambda_0 r) \\ c L r \sin \left( \frac{1}{2} \pi z L^{-1} \right) J_1 (\lambda_0 r) \end{cases} \]

\[ \phi = \begin{cases} \text{0 ≤ r < β} \\ \frac{-1}{2} a_0 r^2 + \theta + a_0 z^2 \\ \frac{-b_0 (\frac{1}{2} r^2 - \ln r) + \theta + b_0 z^2}{\beta < r ≤ 1} \end{cases} \]

\[ T_o = \int \int_{S_o} \phi \vec{u} \cdot \vec{n} \ dS \approx 5.163 \times 10^{-3} L^3 + 0.8 L \geq 0 \]
Conclusions

- Extended Kelvin’s theorem to regions with “open boundaries”
- Kelvin’s theorem holds as long as one of two simple conditions is satisfied
- Identified a new surface decomposition (Kelvin and open boundaries)
- Mass equiflux principle: determine unique potentials
- For flows that were considered, the sufficient condition was always satisfied
- Extension to compressible flows is in preparation